Numbers and Their Application to Math and Science, pdf[4]

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with assistance by

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Berrien County Math & Science Center

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Preface

This web-based series of lectures on *Numbers and Their Application to Math and Science* is the outgrowth of my work since 1993 teaching high school math to a select group of students with very diverse backgrounds. These students attend the Berrien County Math and Science Center at Andrews University. It is assumed, somewhat erroneously, that all students have successfully completed Algebra in eighth grade. All will be taking Geometry or higher in ninth and are expected to complete AP Calculus AB by grade 12.

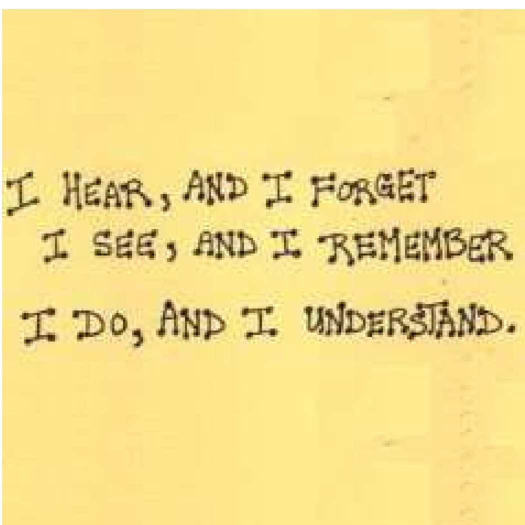
Although these students represent about 1% of the area's high school population, they are well distributed from the top 10% of the rural/small town population across Berrien and Cass Counties. This series of lectures serves to review basic number concepts, apply these concepts to the mathematics and science they will be studying for four years, and also lay a framework for ISEF/EXPO type projects in mathematics, especially during their freshman year. Fundamental concepts essential to doing well on contests, like 0 being even, 1 not being prime, what complex numbers and bases are, are reviewed/taught. Additional purposes include: forcing students to study—many breezed through grade school without cracking a book; separate out the accelerated students (Algebra II, Precalculus); provide a reference booklet for years to come. As such, some material is here for exposure only and not mastry.

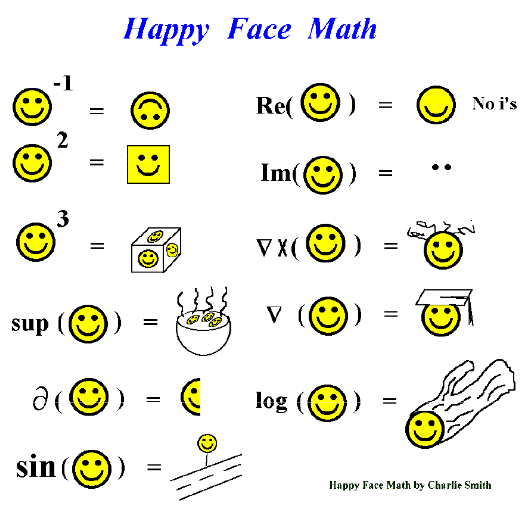
Historically, I was their only math teacher for four years of high school math. This had both benefits and pitfalls. One of the major benefits was the ability to tailor our curriculum's timing and content to the science and technology components of our program. Another was the opportunity to introduce such fundamental concepts of slope, area, bases, proof, *etc.* in such a way as to ease the transition to Calculus. This is still being done by 1) a careful selection and use of a variety of textbooks; 2) different modes of homework usage; 3) selected examples which span a wide variety of subfields of mathematics.

The consolidation of the Berrien County Math and Science Center at Andrews University started in 1992–93, continued in 1997–98, and resulted in a target of 50 students at **one site,** instead of 56–75 per grade level. Expansion to two sections at each grade level was completed in 2000–01, resulting in multiple mathematics teachers. The schedule dictated grade level sections to occur concurrently. Assuring uniform delivery and coverage was also a motivating factor in standarizing this material. However, in 2001–02, we started a return to one section of 30 students per grade level. This resulted in higher SAT scores (freshmen average over 1050) and thus emphasized the need to keep these students challenged.

Numbers are fundamental to the study of mathematics and science. Their discovery (some insist invention) transformed man into rational beings. Concepts such as ratio, continuity, n^{th} roots, significant figures, *etc.* introduced early in our Introduction to Statistics unit stretch the ability of many of our students. This unit is thus designed to complement the instruction given our lowest quartile students in Summer Algebra and somewhat decouple the distraction of these number concepts from the study of Statistics. In addition, students accelerating faster than our normal (and already accelerated) program or those joining late (as Freshmen, Sophomores, and even Juniors or Seniors) need this information which is not well summarized elsewhere.

In 2001–02 we stream-lined the homework by removing some arithmetic and algebra concepts covered in Summer Algebra so it better fits within our 50± minute (55 Tue./Thu.; 45 Fri.) daily class period. In 2006–07 we abandoned the web-page approach and typeset it in book form. We continue to clarify essential concepts and generally improve the delivery.





[1]

I sincerely hope to convey my lifelong passion for numbers as well. I firmly believe mathematics is a "interactive" or participation sport. Although I don't expect to institute cyphering matches (like spelling bee's only doing calculations), lots of other similar activities are planned to involve the students. A Chinese proverb states: "I hear and forget, I see and remember, I do and understand." Understanding is essential for a firm foundation. References such as Google[2] and Dr. Math[3] are also essential.

Sincerely, $Ke^{i\theta}$ or *Keith the Complex number*

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[1] `http://www4.stat.ncsu.edu/~bmasmith/images/all.gif`

[2] `http://www.google.com/`.

[3] `http://www.mathforum.com/dr.math`.

Numbers Lesson 0

0.1 Homework Graded on Day 2

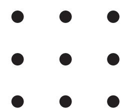
1. Fill in the following table (3×3) with the digits 1–9 (each used and only once) in such a way that each row and column totals 15. You will receive bonus points for **also** having the diagonals so sum.

2. Below is a Sudoku but two by two using the digits 1–4 instead of the more popular three by three version using the digits 1–9. The same rules apply. No number may appear more than once in any given row, column, or two by two smaller box. For one point each digit, complete the Sudoku below.

			2
4			
			1
2			

3. Classify the books in your home six different ways (example hard cover, western, text, *etc.*) and count or estimate how many (what percent) are in each classification. Do they (the percentage) add up to how many books (100%)? If not, why not.

4. A square number, or perfect square, is any number which can be expressed as the product of a number multiplied with itself. For example, $9 = 3 \times 3$, 9 trees can be put into a square figure:



Using the set of digits 1, 6, 9, form as many square numbers as possible. The digits may be reused, such as 11 and 966 (which are not squares), to form larger square numbers. (Bonus for more than five such numbers.)

Outline for Geometry, Wednesday, Sep. 9, 2009

1. 8:00: M–F: Geometry in SH100.
2. Teacher: Keith Calkins, known as Dr. $Ke^{i\theta}$.
3. Pictures: not for publication. Wear name tag ABOVE heart, right-side-up.
4. Introductions: learn everyone's name soon. Learn to speak loudly AND softly.
5. Telephone: share number on list to be redistributed but not published.
6. Notecard: fill out personal information; return TODAY.
7. Textbook: leave at home for reference until mid-October.
8. Numbers: textbook handed out piecemeal. **Do Homework 0 and read Lecture 1 for tomorrow.** Keep all and neat for binding.
9. Notebook: organize notebook with orange notebook check sheet at front. Preferred format is 1" 3-ring binder.
10. Course Outlines: card stock yellow for notebook. Cherry, regular size for home.
11. Syllabus: Get parent to sign ASAP. Math Help Sessions 7–9 pm Tuesdays after first week. Computer Help Sessions 7–9 pm Tuesdays and Wednesdays after first week. Computer helpers are Center graduates or seniors.
12. Calculator: Get TI-nSpire or TI-84+ soon. Bring in proof of purchase seal.
13. Handbooks: Distributed in Computers. Parents sign form.
14. Forms: Turn in forms (medical, handbook, field trip, audio/video), if not done already. Horseplay is not condoned.

Numbers Lesson 1

All About Sets

> *In a small town where all the men are clean-shaven, the barber shaves everyone who does not shave himself. Who shaves the barber?*
>
> Barber's Paradox

In this lesson we will explore the foundations of mathematics, specifically, sets, subsets, and their elements. It is difficult to explain number without this fundamental concept. First, however, we will have the first in our series of biographies of famous mathematicians.

One of the goals of these lectures is to provide familiarity with the great mathematicians. Below we will make reference to Whitehead, Russell, Gödel, Euclid, Pythagoras, Venn, and Euler. In this first lesson we will start with one of the three greatest mathematicians of all time: Archimedes (*c* 287–212 B.C.). (*c.* is an abbreviation for the Latin word *circa*, meaning around.) Newton (1642–1727) and Gauss (1777–1855) will await subsequent lessons. Note that if there were a fourth greatest mathematician, it would be Euler. Learning common Latin (and Greek) terms is another goal.

1.1 One of the Greatest Mathematicians: Archimedes

Archimedes was born, lived, and died in Syracuse, Sicily but studied at Alexandria (Egypt)—at that time the center of learning. He is known as a mathematician, scientist, and inventor, but his greatest contributions were in geometry, such as the relationship between the surface area and volume of a sphere and its circumscribing cylinder. He found lower and upper limits for pi: by inscribing and circumscribing a circle with a regular 96-gon. He invented engines of war (mirrors, catapults, etc.) and the water screw. The principle of bouyancy named after him helped him determine whether or not a crown was pure gold—he streaked from the public bath shouting "*Eureka, Eureka*," or literally I found it, I found it. He is quoted as saying: "Give me a place to stand and I will move the earth"—meaning levers can do great feats. His methods of calculating areas in several cases were equivalent to calculus invented much later. Some of his works were lost and not all the stories and books attributed to him are necessarily his. The author has done extensive research on his cattle problem.[1] Archimedes was drawing geometric figures in the sand when a Roman soldier, approached. Archimedes' last words were: "Do not disturb my circles," when against specific orders, the soldier fatally struck him.

1.2 Sets, Elements, and Subsets

One dictionary has, among the many definitions for set, the following:

Set: a number of things naturally connected by location, formation, or order in time.

Although **set** holds the record as the word with the most dictionary definitions, there are terms mathematicians choose to leave **undefined,** or actually, defined by usage. Set, element, member, and subset are four such terms which will be discussed in today's lesson. Today's activity will also explore the concept of a set.

Each item in or inside a set is termed an **element**.

The brace symbols "{" and "}" are used to enclose the elements in a set.

Each **element** is a **member** of the set (or belongs to the set).

The symbol for membership is "$\in$". It can be read "is an element of" and looks quite similar to the Greek letter epsilon (ϵ). Thus $\epsilon \in \{\alpha, \beta, \gamma, \delta, \epsilon\}$.

A **subset** is a portion of a set.

The symbol for subset is "$\subset$". Some books allow and use it reversed ($\supset$)—we will not.

A **superset** is a set that includes other sets.

For example: If $A \subset B$, then A is a subset of B and B is a superset of A.

A subset might have no members, in which case it is termed the **null set** or **empty set**.

The empty set is denoted either by $\{\}$ or by $\emptyset$, a Norwegian letter. The null set is a subset of every set.

Note: a common mistake is to use $\{\emptyset\}$ to denote the null set. This is actually a set with one element and that element is the null set. Since some people slash their

[1] http://www.andrews.edu/~calkins/profess/cattle.htm.

zeroes, it is safest when handwriting to **always use** the notation $\{\}$ to denote the empty or null set.

A **singleton** is a set with only one element.

A subset might contain every member of the original set.
In this case it is termed an **improper subset**.

A **proper subset** does not contain every member of the original set.

Sets may be **finite**, $\{1, 2, 3, \ldots, 10\}$, or **infinite**, $\{1, 2, 3, \ldots\}$. The **cardinality** of a set A, $n(A)$, is how many elements are in the set. The symbol "…" called **ellipses** means to continue in the indicated pattern. There are 2^n subsets of any set, where n is the set's cardinality.

Example: How many subsets does a set with three elements have?

Solution: $2^3 = 8$. Let the set be $\{A, B, C\}$. Then the subsets are: $\{\}$, $\{A\}$, $\{B\}$, $\{C\}$, $\{A, B\}$, $\{A, C\}$, $\{B, C\}$, and $\{A, B, C\}$. We will discuss the pattern made by the number of subsets of each cardinality in a later lesson.[2]

The **power set** of a set is the complete set of subsets of the set.

For any set its power set is at least as big, if not bigger than the original set. That is, $2^n > n$ for all $n \geq 0$. We will have reason to explore this later when we discuss levels of infinity.

Example: for the set $\{A, B, C\}$, the power set would be:
$\Big\{\{\}, \{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}, \{A, B, C\}\Big\}$.

In this class we will consider only safe sets, that is, any set we consider should be **well-defined**. There should be no ambiguity as to whether or not an element belongs to a set. That is why we will avoid things like the village barber who shaves everyone in the village that does not shave himself. This results in a contradiction as to whether or not he shaves himself. See also Titus 1:12 in the Bible: "A Cretan said: all Cretans are liars." Also consider Russell's Paradox: Form the set of sets that are not members of themselves. It is both true and false that this set must contain itself. These are examples of **ill-defined** sets.

Sometimes, instead of listing elements in a set, we use **set builder** notation: $\{x \mid x \text{ is a letter in the word "mathematics"}\}$. The symbol "|" can be read as "such that." Sometimes the symbol "$\subset$" is reserved to mean proper subset and the symbol "$\subseteq$" is used to allow the inclusion of the improper subset. Compare this with the use of $<$ and $\leq$ in Section 9.4 to exclude or include an endpoint. We will make no such distinction. A set may contain the same elements as another set. Such sets are **equal** or **identical** sets—element order is unimportant. $A = B$ where $A = \{m, o, r, e\}$ and $B = \{r, o, m, e\}$, in general $A = B$ if $A \subset B$ and $B \subset A$. Sets may be termed

[2]Hint: use Pascal's Triangle.

equivalent if they have the same cardinality. If they are equivalent, a **one-to-one correspondence** can be established between their elements.

The **universal set** is chosen arbitrarily, but must be large enough to include all elements of all sets under discussion.

Complementary set, A' or $\overline{A}$, is a set that contains all the elements of the universal set that are not included in A. The symbol "′" can be read "prime."

For example: if $U = \{0, 1, 2, 3, \ldots\}$ and $A = \{0, 2, 4, \ldots\}$, then $A' = \{1, 3, 5, \ldots\}$.

Such paradoxes as those mentioned above, particularily involving infinities (discussed in the next lesson), were well known by the ancient Greeks. During the 19^{th} century, mathematicians were able to tame such paradoxes and about the turn of the 20^{th} century Whitehead and Russell started an overly ambitious project to carefully codify mathematics. Set theory was developed about this time and serves to unify the many branches of mathematics. Although in 1931 Kurt Gödel showed this approach to be fatally flawed, it is still a good way to explore areas of mathematics such as: arithmetic, number theory, [abstract] algebra, geometry, probability, *etc.*

Geometry has a long history of such systematic study. The ancient Greek Euclid similarily codified the mathematics of his time into 13 books called *The Elements*. Although these books were not limited to Geometry, that is what they are best known for. In fact, up until about my grandfather's day, *The Elements* was the textbook of choice for the study of Geometry! *The Elements* carefully separated the assumptions and definitions from what was to be proved. The concept of proof dates back another couple hundred years to the ancient Greek Pythagoras and his school, the Pythagorean School.

1.3 Intersection and Union

Once we have created the concept of a set, we can manipulate sets in useful ways termed **set operations**. Consider the following sets: animals, birds, and white things. Some animals are white: polar bears, mountain goats, big horn sheep, for example. Some birds are white: dove, stork, sea gulls. Some white things are not birds or animal (but birds are animals!): snow, milk, wedding gowns (usually).

The **intersection** of sets are those elements which belong to all intersected sets.

Although we usually intersect only two sets, the definition above is general. The symbol for intersection is "∩".

The **union** of sets are those elements which belong to any set in the union.

Again, although we usually form the union of only two sets, the definition above is general. The symbol for union is "∪".

For the example given above, we can see that:
$\{\text{white things}\} \cap \{\text{birds}\} = \{\text{white birds}\}$
$\{\text{white animals}\} \cup \{\text{birds}\} = \{\text{white animals and all birds}\}$
$\{\text{white birds}\} \subset \{\text{white animals}\} \subset \{\text{animals}\}$

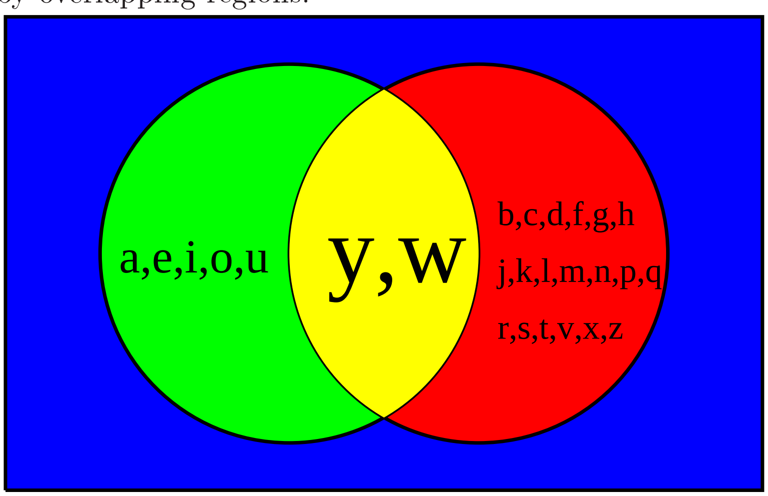
Another name for intersection is **conjuction**. This comes from the fact that an element must be a member of set A **and** set B to be a member of $A \cap B$. Another name for union is **disjunction**. This comes from the fact that an element must be a member of set A **or** set B to be a member of $A \cup B$. Conjunction and disjunction are grammar terms and date back to when Latin was widely used.

I should note the very mathematical use of the word **or** in the sentence above. Common usage now of the word or means one or the other, but not both (excludes both). Mathematicians and computer scientists on the other hand mean one or the other, possibly both (including both). This ambiguity can cause all kinds of problems! Mathematicians term the former exclusive or (EOR or XOR) and the latter inclusive or. We will see ands & ors again in Numbers Lesson 7 on truth tables.

1.4 Pictures of Sets (Euler/Venn Diagrams)

John Venn (1834–1923) extended the use of diagrams first developed by Leonhard Euler (1707–1783), the great Swiss mathematician, to give pictures of sets. Venn diagrams are often used to visualize set operations.

A superset does not have to be the universal set. The above example has white things as a superset of white birds, while the set containing both animate and inanimate objects is another possible universal set. A rectangle should be used to enclose the universal set, and other sets under discussion are enclosed inside. Relationships are indicated by overlapping regions.



Here, the English alphabet is our universal set. Vowels and consonants are nondisjoint subsets thereof. **Disjoint** would mean their intersection was empty.

1.5 List of Greek/Latin Terms

Several different Greek and Latin terms and other abbreviations are purposefully used in this series of lectures. Most are listed here for reference.

- See Sec. 1.6: *aka*, also known as
- See Sec. 1: *c*, *circa*, around
- See Sec. 2.3 *cf*, *confer*, compare
- See Sec. 6.1 *Cogito ergo sum*, I think, therefore I am.
- *eg*, *exempli gratia*, for example
- See Sec. 2.3: *etc.*, *et cetera*, and so forth
- See Sec. 1: *i.e.*, *id est*, that is (to say)
- juxtaposition, placed side-by-side
- See Sec. 10.5: *lb*, *libra*, pounds (weight), scales
- See Sec. 10.4 and 15.4: *mantissa*, mantissa, makeweight
- See Sec. 2.10: *mod*, *modulo*, a small measure
- See Sec. 6.4: *Modus Ponens*, Law of Detachment.
- See Sec. 6.4: *Modus Tollens*, Law of indirect reasoning.
- *nb*, *nota bene*, note well
- See Sec. 11.4: Q.E.D., *quod erat demonstrandum*, that which was to be shown/demonstrated
- See Sec. 8.7: *vice versa*, order opposite
- See Sec. 15.1: *viz*, *videlicet*, namely
- See Sec. 10.2: *vs*, *versus*, against or facing

1.6 Set Homework

This homework was originally designed to motivate some lecture topics and set up some information for later reference (problems 1–7). Also, it can take a week or more for such matters as buying or borrowing a graphing calculator to be resolved. Each problem is worth two homework points unless otherwise noted.

1. (4 points) Count to ten by ones.
   (a) Write these numbers down in order both with names (words) and in symbols (digits).
   (b) What number did you start with? Why?
   (c) What number comes **next** after ten?
   (d) How many numbers come before ten?

2. (3 points) Suppose you have two rectangular egg cartons each filled with a dozen eggs. However, the egg cartons are not the same shape–*i.e.* one is long and skinny, the other is short and fat. (*i.e.* is an abbreviation for the Latin term *id est* meaning **that is (to say)**.)
   (a) What are the two most likely configurations of eggs in these cartons?
   (b) What is another possible, but unlikely configuration?
   (c) What are **two** ways to show that each carton has the same number of eggs?

3. Repeat problem 1, part (a), but instead of assuming **Arabic numbers**, write your results using **Roman numerals** (no words needed).

4. Begin with the number two (in Arabic numerals).
   - Double the current number either by multiplying by two or adding itself.
   - Repeat this process a total of ten times. Be sure to show your work.

5. Suppose a new toy costs a hundred clams, but you only have eighty-nine clams. After you buy the toy, how many clams do you have (*i.e.* you may have borrowed)? Show your work.

6. By long division and showing your work, determine how many times six goes into one million. If it did not go evenly, what is the remainder?

7. Preferably using the process of long division and showing your work, determine how many times seven goes into one million. If it did not go evenly, what is the remainder?

8. Name a counting song. (Consider bringing it, if really special.)

For problems 9–11: Given $A = \{m, a, t, h\}$ and $B = \{e, a, s, y\}$.

9. Find $A \cup B$.

10. Find $A \cap B$.

11. Find A' (also known as (aka): $\overline{A}$).

12. (3 points) Are these statements true or false. Venn diagrams may be helpful.
    (a) $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$?
    (b) $(A \cup B) \cup C = A \cup (B \cup C)$?
    (c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$?

13. (Future test points) In your Geometry textbook, read section 2.5 and look carefully at problems 5–10, 15, and 16. Note the application of unions and intersections to geometric figures.

14. (0 points) Learn the game of Set® and prepare for a double elimination Set® tournament!

Numbers Lesson 2

God Invented The Integers

God invented the integers. All the rest is the work of man. Kronecker

There is a philosophic question as to whether man discovered or invented numbers. This lesson title and quote are part of that debate. Although we discuss the natural numbers in this lesson, we defer into the next lesson the development of them using the Peano axioms and mathematical induction. Of course, there is an important choice as to where to start: zero or one. Where to stop is another important question! Alternate methods of developing sequences are noted which lead to Triangular Numbers, Fibonacci Numbers, the Integer, and Factorials. Integer division (an integer divided by an integer yielding an integer quotient and an integer remainder) is discussed. First we discuss a second great mathematician.

2.1 One of the Greatest Mathematicians: Gauss

Johann Carl Friedrich Gauss was German, born the only son of poor parents. However, his early genius was recognized as discussed later in this lesson at a young age. In his doctoral thesis at age 22, he developed the concept of complex numbers and the Fundamental Theorem of Algebra. He applied mathematics to gravitation, electricity, and magnetism, thus his name is closely tied into modern physics. Some of his important quotes are "Mathematics, the queen of the sciences, and arithmetic, the queen of mathematics" and "*Pauca, sed matura* (few, but ripe)." Gauss is perhaps most famous for what I like to rather redundantly call the bell-shaped, gaussian, normal curve which we will study later. He is also known for his method of least squares to obtain the best regression line which we will study much later.

2.2 The Development of Mathematics via Axioms, Definitions, and Proof

An **axiom** is a statement assumed to be true.

Postulate is another word for axiom.

Axioms and logical reasoning together enable mathematicians to prove things. In this section we will present and discuss certain axioms from which all the properties of the natural numbers may be proved. Later lessons will develop the concept of logical reasoning and **proof**. First, we will present groups of axioms to help us understand the different number systems we will encounter.

Undefined words in today's lesson include the following: equal, successor, and number. The terms addition, multiplication, subtraction, and division will also not be rigorously defined, but must satisfy the group and field axioms presented in Lesson 8 and lesson 14. You were taught rudimentary **algorithms** in grade school which we will review very briefly.

2.3 Natural or Counting Numbers and Whole Numbers

The **natural** or **counting numbers** are the familiar set: $1, 2, 3, 4, 5, \ldots$

The ellipses symbol $\ldots$ (often read as dot dot dot) is often abbreviated *etc.*, which is an abbreviation for the Latin term *et cetera* meaning **and so forth**.

There is actually no uniform agreement as to whether or not zero (0) is a natural number. Popular usage indicates that it is not, whereas books on number theory will often define it to be one! Computer scientists and some popular programming languages such as C and C^{++} also often treat it as a counting number. The difference can be summarized by where we point or index (*cf* your index finger). *cf* is an abbreviation for the Latin *confer* meaning **compare**.

Most books define **whole numbers** as the union of the counting numbers with zero.

2.4 Zero and One Indexing

Zero Indexing acknowledges zero as the number we start counting with.
One Indexing acknowledges one as the number we start counting with.

In this class we will be flexible, but try to specify when zero indexing is to be used.

The symbol $\mathbb{N}$ or $\mathcal{N}$ is used to denote the set of natural numbers.

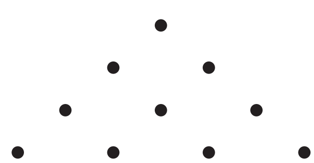
2.5 The Countable Infinity, $\aleph_0$

The set of natural numbers is an infinite set. There is always a next larger number. The perhaps misguided concept of "biggest number" is usually conveyed by the term infinity and symbol ∞. Actually, this symbol is most commonly used in the context of the real numbers. For integers, the symbol $\aleph_0$ is commonly used. $\aleph$ is the first letter of the Hebrew alphabet and is called aleph, much like α or alpha. The subscript is usually termed null instead of zero, hence aleph-null. The concepts of infinity, infintesimal, and continuity were the root cause of several ancient Greek paradoxes which we will explore further in Lesson 14.

2.6 Addition and Triangular Numbers

When we **add** two numbers together, they are termed **addends**. The result is termed the **sum**.

An interesting subset of the natural numbers generated by addition are called Triangular Numbers. These are so called because these are the total number of dots, if we arrange the dots in a triangle with one additional dot in each layer.



The triangular numbers thus are: $0, 1, 3, 6, 10, 15, 21, \ldots$ (Not everyone considers 0 to be triangular.)

The following example has a rich history dating back to the early childhood of Gauss. To keep his class busy for a long time, the teacher told them to add the counting numbers up to one hundred. Gauss finished very quickly thus revealing his early genius. This is what he did:

$$T_{100} = (1 + 100) + (2 + 99) + (3 + 98) + \ldots + (50 + 51) \quad (2.1)$$
$$= 101 \times 50 \quad (2.2)$$
$$= 101 \times \frac{100}{2} \quad (2.3)$$

Note how the equal signs are aligned vertically, a form we will strongly encourage to reduce mistakes.

This can be generalized to: $T_n = \sum_1^n i = \frac{n(n+1)}{2}$, where mathematicians use the capital Greek letter $\sum$ (sigma) to represent **summation**. One of your teachers has a particular fondness for this symbol since the first computer he had much access to had that nickname.

2.7 Fibonacci Numbers

Another way to add numbers together generates the Fibonacci Numbers. A biography for this early Italian mathematician will come in a later lesson.[1] Historically, this sequence was associated with the number of progeny a pair of rabbits produced given a month to mature and a monthly reproductive cycle. However, it appears in such diverse places as sunflower spirals and 3" by 5" cards.

Fibonacci Numbers, represented here by L_i, can be defined as follows.
Let $L_0 = 0$ and $L_1 = 1$. For all other L_i, let $L_{n+1} = L_{n-1} + L_n$.

This definition is **recursive**,[2] *i.e.* each term is defined in terms of the previous two. The first few Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8, 13, ... (Not everyone considers 0 to be a Fibonacci number.)

2.8 Factorials

We **multiply** a **multiplicand** by a **multiplier**, and call the result a **product**.

Factorials can be defined recursively as $n! = n \times (n-1)!$ where $1! = 1$.

By definition, 0!=1. (Don't ask, it just works best!)

For example, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 5 \times 4 \times 3 \times 2 = 5 \times 4 \times 6 = 5 \times 24 = 120$.

In general, $n! = \prod_{i=1}^n i$. The symbol Π is the capital Greek letter pi (π) and represents **product**. The expression is termed a **pi product**.

[1] See `http://www.engineering.sdstate.edu/~fib` for more information.

[2] We used the symbol L in honor of Fibonacci's first name Leonardo, for the general Lyman sequences of which the Fibonacci sequence is most famous, and to avoid confusion with Fermat numbers.

2.9 Subtraction and the Rest of the Integers

Early in life, most of us encounter negative numbers, for example, when something costs more than what we have. Perhaps, we are able to get an advance on our allowance and thus encounter debt.

When you **subtract** a **subtrahend** from a **minuend**, the result is termed the **difference**.

The **integers** are the counting numbers together with their opposites and zero.

Opposite in this case refers to the concept of additive inverse (a field axiom). It would seem that we have doubled the size of the number system, but in actuality it is still a countably infinite set.

The symbol $\mathbb{Z}$ or $\mathcal{Z}$ is used to denote the set of integers.
It comes from the German word zahlen, meaning to count.

2.10 Integer Division or Division with Remainder, Modulo, Congruence

Even: An integer is even if it is an integer multiple of 2.

Odd: An integer is odd if it is not an integer multiple of 2.

Hence, the even numbers are $0, \pm 2, \pm 4, \ldots$ and the odd numbers are $\pm 1, \pm 3, \pm 5, \ldots$ Zero **is** even.

Although division will be presented again later, a special form will be introduced here. Often the remainder obtained in a division is more important than the quotient.

When a **dividend** is **divided** by a **divisor**, the results are termed the **quotient** and **remainder**, where quotient is the number of times the divisor went into the dividend and the remainder is how many were left over.

When doing long division, it looks like this: $\text{Divisor}\,\overline{)\text{Dividend}}^{\text{Quotient R Remainder}}$.

The concept of even and odd introduced above can be expressed as whether the remainder was 0 or 1 when divided by 2. This can be expressed as $0 \bmod 2$ or $1 \bmod 2$ where mod is an abbreviation for the Latin term *modulo* meaning **a small measure**. The same syntax is often used to ask the question: What is $121 \bmod 2$? Answer: 121 is $1 \bmod 2$, or an odd number. We also say, $121 \equiv 1 \pmod 2$ Where $\equiv$ is read equivalent to. A later homework problem will extend this concept to your every day experience such as telling time.

Modulo is the remainder when dividing by a divisor.

Numbers which have the same remainder when divided by another are termed **congruent**. Congruence will have other uses in geometry to indicate two objects have both the same shape and measure.

2.11 Counting Homework

Each problem is worth three points.

1. Complete the following addition table.

+	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
0													
1													
2													
3													
4													
5													
6													
7													
8													
9													
10													
11													
12													

2. Use the information above to complete the following table about even and odd numbers. Even or odd should be used to fill in the blanks.

+	even	odd
even		
odd		

3. Write out the first 15 Fibonacci Numbers.

4. Consider each Fibonacci Number as either even or odd. What is the pattern? How does this follow from the above even/odd addition table?

5. Find up to five Fibonacci Numbers which are Triangular Numbers.

6. Find six numbers which satisfy the expression (are congruent to): 1 mod 5.

7. Find the sum of all the integers from 1 to 50, inclusive.

8. Complete the following multiplication table.

×	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
0													
1													
2													
3													
4													
5													
6													
7													
8													
9													
10													
11													
12													

9. Use the information above to complete the following table about even and odd numbers.

×	even	odd
even		
odd		

10. **Bonus points: (An easy version of a Fibonacci classic)** A snail landed at the bottom of a 30 foot well. It climbs up 3 feet every day, but slides back down 2 feet each night. How long will it take the snail to get out of the well?

Numbers Lesson 3

The Peano Axioms

As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.

Albert Einstein

This lesson allows us to slow down here in the early lessons and take a look at how the counting numbers are developed (Peano Axioms), make reference to a couple controversial axioms (Well-ordered, and Least Cardinal), and list the mathematicians we well be studying.

3.1 Father of Geography: Eratosthenes

Eratosthenes was a Greek mathematician, poet, athlete, geographer, and astronomer (276–194 B.C.) In mathematics, he is perhaps best known for his sieve algorithm for obtaining prime numbers which bears his name and is developed in the next lesson.

Eratosthenes made remarkable discoveries, inventions, and measurements. For example, he developed the system of latitude and longitude, first calculated the circumference of the earth, tilt of the earth's axis, the earth-sun distance, and invented the leap day. His contemporaries nicknamed him "Beta," meaning two or second, because he was second best, but in so many different fields.

Eratosthenes was the chief librarian of the Great Library in Alexandria, studied for a time in Athens, and was a friend to Archimedes.

3.2 Dedekind-Peano Axioms and Mathematical Induction

- 1 is a member of the set $\mathbb{N}$.
- If n is a member of $\mathbb{N}$, then $n+1$ belongs to $\mathbb{N}$ (where $n+1$ is the "successor" of n.
- 1 is not the successor of any element in $\mathbb{N}$.
- If $n+1 = m+1$, then $n = m$.
- A subset of $\mathbb{N}$ which contains 1, and which contains $n+1$ whenever it contains n, must equal $\mathbb{N}$.

In general, we don't emphasize the above axioms in this class, but they are presented here to assure you the natural numbers were discovered, exist, and/or can be created (just in case you had any doubt). Some additional Peano Axioms are listed in Lesson 14.4.

Axiom 5 above is the basis for **mathematical induction** which will be developed later (Geometry, Chapter 11).

3.3 Well-Ordering Axiom

Well-Ordering Axiom: Any nonempty set of positive integers contains a least element.

The **minimum** is another term for least element. The largest element is the **maximum**. An important note to remember is that the integers do have an order (but no minimum or maximum)! Also, the Well-Ordering Axiom is at the center of some controversy. It is equivalent to the Axiom of Choice and thus the root of the Continuum Hypothesis. See Numbers lesson 14 for more details.

3.4 Cardinal *vs.* Ordinal Numbers

Cardinal Numbers are positive integers (counting numbers) that represent "how many?"

Ordinal Numbers are numbers that describe position: first, second, third, fourth,... last

An example: There are nine innings ("how many?") in a baseball game. Right now in the ninth inning (position), there is a man on first and third with two outs. We also saw the term cardinality in section 1.2 where it was used to indicate the size of a set, as in how many elements a set had.

In this last context, the cardinality or size of a set, is where controversy arises. It is well known that not all infinite sets are the same size and thus there arises a heirarchy of cardinals, possibly well-ordered. This relates to the continuum hypothesis and a host of related axiom proposals which some think should be accessible to the gifted high schools student.

3.5 List of Mathematicians

Many different mathematicians are referenced in this series of lectures. There are two lists provided here. First are those for whom a short biography is provided and for which the student should make a conscious effort to learn about this semester. Freshmen will do presentations about these mathematicians during the second nine week period. The second list is of those of a more incidental nature whose names are attached to an important concept and the concept should be learned. Sophomores will do presentations about these mathematicians in their fall semester.

3.5.1 Mathematicians/Scientists with Short Biographies

- Sec. 1.1: Archimedes (*c.* 287–212 B.C.), one of greatest mathematicians/physicists.
- Sec. 9.1: Georg Cantor (1845–1918), set theory, transfinite numbers.
- Sec. 16.1: Abraham de Moivre, (1667–1754), complex root finding theorem.
- Sec. 6.1: René Descartes (1591–1650), French, analytic Geometry.
- Sec. 3.1 and 4.6: Eratothenes (about 200 B.C.), Greek, prime sieve, earth's circumference.
- Sec. 7.1: Euclid (about 300 B.C.), Greek, Father of Geometry, Even Perfects.
- Sec. 13.1: Leonard Euler (1707–1783), $(2^{2^5}+1)/641$=integer

- Sec. 5.1: Pierre de Fermat amateur mathematician, early 1600's, $2^{2^n} + 1$, $x^n + y^n \neq z^n, n > 2 =$ FLT.
- Sec. 12.1: Fibonacci, 13th century Italian; $0, 1, 1, 2, 3, 5, \cdots$; rabbits, arabic algorithms.
- Sec. 2.1: Carl Friedrich Gauss (1777–1855), one of greatest mathematicians/physicists.
- Sec. 7.3 and 14.1: Kurt Gödel (1906–1978), 1931 Gödel's Incompleteness Theorem.
- Sec. 15.6: David Hilbert (1862–1943), 23 problems of 1900, *Foundations of Geometry*.
- Sec. 4.8 and 8.1: Marin Mersenne, (1588–1648), French monk, numbers/primes of form $2^n - 1$.
- Sec. 15.1: John Napier, (1550–1617), Scotland, logs, slide rule, decimal point.
- Sec. 4.1: Sir Isaac Newton (1642–1727), invented calculus, three laws of motion, universal gravitation, one of greatest mathematicians/physicists.
- Sec. 5.10 and 10.1: Blaise Pascal (1623–1662), triangle, pressure gauge, calculator.
- Sec. 11.1: Pythagoras (*c.* 500 B.C.), Greek school, $a^2 + b^2 = c^2$ iff $\triangle ABC$ is right.

3.5.2 Mathematicians Noted More in Passing

- Sec. 14.3: Niels Henrik Abel (1802–1829), abelian=commutative.
- Sec. 7.3: George Boole (1815–1864), Boolean Algebra.
- Sec. 15.4: Henry Briggs (1561–1631), log tables.
- Sec. 14.1: Paul Cohen (1934–present), 1963 showed independence of CH and AC.
- Sec. 14.7: John Conway, (1937–present), surreal numbers, game of life.
- Sec. 15.2: Richard Dedekind (1831–1916), German, Dedekind Cut defines real numbers.
- Sec. 7.3: Augustus De Morgan (1806–1871), DeMorgan's Law.
- Sec. 12.6: Diophantus of Alexandria (about 250 A.D.), integer solutions.

- Sec. 12.10: Christian Goldbach (1690–1764), conjecture: all evens=sum of two primes.
- Sec. 15.4:: Johannes Kepler (1571–1630), three laws of planetary motion.
- Sec. 14.7: Donald E. Knuth (1938–present), TeX, LaTeX, MetaFONT, Art of CP.
- Sec. 4.1: Leibnitz, (1646–1716), German, coinventor of calculus.
- Sec. 3.2: Guiseppe Peano (1858–1932), Axioms, induced the natural numbers.
- Sec. 1.4: John Venn (1834–1923), set union/intersection diagrams.
- Sec. 12.6: Andrew Wiles (1953–), proved Fermat's Last Theorem.

3.6 Peano Homework

Each problem is worth 3 points.

1. Given $A = \{-2, 0, 4, 7\}$ and $B = \{-4, -2, 0\}$, show both $A \cup B$ and $A \cap B$ using Venn diagrams.

2. Given $A = \{x \mid x > 4\}$ and $B = \{x \mid x < 3\}$, find $A \cup B$ and $A \cap B$ using real number lines.

3. Given $M = \{\text{residents of Michigan}\}$ and $N = \{\text{residents of Niles, Michigan}\}$, describe in words $M \cup N$ and $M \cap N$.

4. Given $B = \{\text{youths attending BCYF}\}$ and $C = \{\text{BCM\&SC students}\}$, describe in words $B \cup C$ and $B \cap C$.

5. Draw a Venn diagram for the previous exercise. What might the Universal set be?

6. Given: $X = \{1, 3, 5, 7, 9\}$, $Y = \{1, 6, 11, 16, \ldots\}$, $Z = \{0, 2, 4, 6, 8, \ldots\}$. Find:
   (a) $(X \cap Z) \cap Y$
   (b) $(X \cup Y) \cap Z$

7. Simplify exactly: a. $9!$ b. $6! \div 3!$ c. $8! \times 8! \div (10! \times 5!)$

8. Evaluate the sum of the following: a. $\sum_{k=0}^{5}(k+2)$ b. $\sum_{k=2}^{4}(2k+3)$

9. Calculate the powers of 11 from 11^0 up to 11^6. Write each one centered below the previous one.

10. Examine the factors of 231 and express it in a form relating it to the triangular number formula.

Numbers Lesson 4

The Naturals as Prime or Composite

Chebychev said it and I'll say it again,
There's always a prime between n and $2n$! Nathan Fine

The natural numbers have been studied intensely for millenia. Several fascinating properties relate to their factors. We will explore these properties such as number of factors and sum of factors in this lesson.

4.1 One of the Greatest Mathematicians: Newton

Sir Isaac Newton, tiny, weak, and not expected to survive his first day, was born in England on Christmas day (old style) 1642. He is known not only as one of the greatest mathematicians, but also one of the greatest physicists as well. He culminated (to climax) the scientific revolution and authored *Principia*, the most important single work in the history of modern science. Newton attended Trinity College, then laid the foundation of calculus and extended his ideas on color. He examined planetary motion and derived the inverse square law crucial to his theory of universal gravitation. The three laws of mechanics were named after him. He was also warden, then later master, of the mint. There he oversaw a great recoinage which included reeded edges on coins and tracking down a master counterfeiter. Two important quotes attributed to Newton are "If I have seen a little farther than others it is because I have stood on the shoulders of giants" and "I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me."

Returning home from work at the Mint, Newton solved a mathematical problem that was given to European mathematicians to solve; he turned in his work the next day anonymously. Upon receiving the solution, John Bernoulli exclaimed, "Ah! I recognize the lion by his paw." Newton was knighted for his scientific discoveries rather than deeds on the battlefield—a first. Newton was buried like a king in Westminster Abbey. Late in Newton's life a battle raged between the English and Germans regarding whether Newton was the sole inventor of calculus or if Leibnitz had also played an important role.

4.2 Factors, Prime, Composite, 1 is Unique

A **factor** is a natural number which divides another natural number evenly (as in without a remainder).

The word factor will be used later in a less restricted sense as in $x-1$ and $x+1$ are factors of x^2-1. **Divisor** is essentially a synonym of factor and is also commonly used interchangeably.

A **prime number** only has factors of itself and one.

The first few prime numbers are: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47....

Twin primes are primes which differ by 2.

Examples of twin primes are: 3 and 5, 5 and 7, 11 and 13, 17 and 19, ... The twin prime conjecture states there are an infinite number of twin primes. It is believed to be true but a recent proof was found flawed.

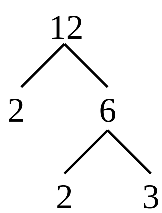
A **composite number** has factors in addition to itself and one.

One (1) is unique in that it is considered neither prime nor composite.

Example: The number 12 has the following factors: 1, 2, 3, 4, 6, and 12. A number such as 12 can also be factored into prime factors: $12 = 2^2 \times 3^1$. For integers, if arranged in order, such factoring is unique.

A **prime factor** is a factor that is prime.

There is a relationship between the prime factors and the number of factors; it involves the exponents. We will examine this in the homework.



A **factor tree** is a common way to find factors and I'm sure a TI-84+ calculator program is also floating around. An example of a factor tree is given to the left.

Example: Consider factoring 180 and 210. There are a wide variety of ways to construct a factor tree, but the final factorization remains the same.

Solution: $180 = 10 \cdot 18 = 2 \cdot 5 \cdot 2 \cdot 3^2 = 2^2 \cdot 3^2 \cdot 5$ and $210 = 10 \cdot 21 = 2 \cdot 5 \cdot 3 \cdot 7 = 2 \cdot 3 \cdot 5 \cdot 7$.

4.3 Prime Factorization, GCF, LCM

Once a natural number has been factored into prime factors, we can write its prime factorization (also known as prime decomposition). When we do this, we list each prime factor in **increasing order** and indicate how many times it is repeated by **using a superscript as an exponent**. For example: $60 = 2^2 \times 3^1 \times 5^1$. When done this way, the prime factorization for the natural numbers is unique. The associated prime factorization theorem (or Fundamental Theorem of Arithmetic) could be proved, but not here.

We can use prime factorization to find Greatest Common Factors and Least Common Multiples. Another method is **Euclid's Algorithm** (a procedure) which we intend to link to here eventually.

GCF: Greatest Common Factor (or GCD) is the greatest number that divides two given numbers.

Example: The factors of 30 are {1, 2, 3, 5, 6, 10, 15, 30} and the factors of 12 are {1, 2, 3, 4, 6, 12} and so the factors 30 and 12 have in common are {1, 2, 3, 6}. The GCF would then be 6.

Two numbers are **relatively prime** if they have no common factors (excluding 1).

In other words, two numbers are relatively prime if their GCF is 1. Examples are: 15 and 16, 20 and 21.

LCM: Least Common Multiple is the smallest (positive) number which is a multiple of two numbers.

The definitions of GCF and LCM could be extended to more than two numbers. In fact, since the calculator will only do pairs, such an extension gives more meaningful test questions!

Example: The multiples of 4 are: {4, 8, 12, 16,...} and 6 has multiples of {6, 12, 18, 24, 30, ...}. The intersection of these sets is {12, 24, 36...}, so the LCM is 12.

Example (Using Prime Factorization): $30 = 2^1 \times 3^1 \times 5^1$ and $12 = 2^2 \times 3^1$. Thus the GCF(12, 30) is $2^1 \times 3^1 = 6$ and the LCM(12, 30) is $2^2 \times 3^1 \times 5^1 = 60$. Notice how for GCF we choose the smallest exponent for each prime factor and for LCM we choose the largest. It might help to note that $12 = 2^2 \times 3^1 \times 5^0$ and remember that anything to the zero power is 1. Note how $\text{GCF}(12, 30) \times \text{LCM}(12, 30) = 6 \times 60 = 12 \times 30$.

Example: $25 = 5^2 \times 17^0$ and $85 = 5^1 \times 17^1$. The $\text{GCF}(25, 85)$ is $5^1 \times 17^0 = 5$ (choosing the smallest exponents) and the $\text{LCM}(25, 85)$ is $5^2 \times 17^1 = 425$ (choosing the largest exponents).

4.4 Number of Factors

We can tell how many factors a number has using only the exponents from its prime factorization. Suppose $p_1^{q_1} \cdot p_2^{q_2} \cdot p_3^{q_3}$ is the prime factorization of some number N. There are $(q_1+1)(q_2+1)(q_3+1)$ factors since each p_i to all powers from 0 to q_i and whether or not each prime is a factor is independent.

Example: $180 = 2^2 \cdot 3^2 \cdot 5^1$, There are $3 \cdot 3 \cdot 2 = 18$ factors, namely: $\{1, 180, 2, 90, 3, 60, 4, 45, 5, 36, 6, 30, 9, 20, 10, 18, 12, 15\}$.

Example: $210 = 2 \cdot 3 \cdot 5 \cdot 7$. There are thus 2^4 factors of 210.

4.5 Primes Form an Infinite Set

It can easily be shown that the set of prime numbers is infinite. This proof, which dates back to Euclid, (link) is as follows. Suppose, on the contrary, that there are only finitely many primes denoted $p_1, p_2, \ldots p_n$. Form the product $N = p_1 \times p_2 \times p_3 \times \ldots \times p_n$. Then, the number $N+1$ is not divisible by any p_i and so must be divisible by a prime other than these (including possibly only $N+1$ itself). This contradicts our original hypothesis that we listed all the (finite set of) primes, hence this hypothesis is false. Hence there must be infinitely many primes. This is a classic proof by contradiction. It remains an open question whether or not there are an infinite number of twin primes. Using the well-ordering axiom, we can also prove all numbers are interesting!

4.6 Sieve of Eratosthenes

Having established the fact that there are infinitely many primes, we might want to generate a list of primes, or determine if a given number is prime. Eratosthenes, a Greek mathematician around 200 B.C., created a simple algorithm[1] to find primes. The procedure represents a sieve, or device used for sifting out grains, since he actually punched holes. The method is simple:

1. Write down the numbers from 1 to 100 (or any desired range).
2. Start with two (the first prime number).
3. Eliminate all its multiples.
4. Move to the next prime (the next number on the list which you have not eliminated).
5. Go back to step 3 and repeat as many times as necessary.

[1] `http://en.wikipedia.org/wiki/Eratosthenes` has a link to a java script.

Note that anything above $\sqrt{100} = 10$ does not eliminate any more numbers, since factors come in pairs of a big and a small.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

4.7 Division Rules

Here are some useful rules for quickly checking for divisibility of natural numbers by small factors.

Divisibility by 2: If an integer is even, that is ends in 0, 2, 4, 6, or 8, it is divisible by 2.

Divisibility by 3: If the sum of the digits of an integer is divisible by 3, then the integer is divisible by 3.

Example: $729 \rightarrow 7 + 2 + 9 = 18 \rightarrow 1 + 8 = 9$. Thus 729 is divisible by 3. Note how this was done recursively.

Divisible by 4: If the last two digits of the integer are divisible by 4, then the integer is divisible by 4.

In general, an integer is divisible by 2^n if the last n digits are divisible by 2^n.

Divisibility by 5: If the last digit is 0 or 5, the integer is divisible by 5.

If the last n digits are divisible by 5^n, then the integer is divisible by 5^n.

Divisibility by 9: If the sum of the digits of an integer is divisible by 9, then the number is divisible by 9.

A common method taught in days past for finding computational mistakes was called Casting Out 9. This is really a form of *modulo* arithmetic. In other bases, this method extends to "Casting Out base - 1."

Divisibility by 11: If the sum of the digits in the even powers of 10 positions differ from the sum of the digits in the odd powers of 10 positions by a multiple of 11, the integer is divisible by 11.

Example: $1,234,508 \rightarrow 1+3+5+8 = 17$ and $2+4+0 = 6$, thus since $17-6 = 11$, 1,234,508 is divisible by 11.

In general, determining if a large number is prime or composite is a difficult task. Substantial research continues in this field due to the fact that many encryption schemes are dependent on this difficulty.

4.8 Perfect Numbers and Mersenne Primes

A **perfect number** is equal to the sum of its factors, excluding itself.

The first two perfect numbers are:
$6 = 1+2+3 = 1 \times 6 = 2 \times 3 = 2^{2-1} \times (2^2 - 1)$ and
$28 = 1+2+4+7+14 = 1 \times 28 = 2 \times 14 = 4 \times 7 = 2^{3-1} \cdot (2^3 - 1)$.

The ancients considered these numbers perfect partly due to their close proximity to the number of days in a week (which is not celestial!) and the lunar/menstral cycle.

Mersenne Numbers are of the form $2^n - 1$.

Mersenne Primes are primes of the form $2^n - 1$.

A biography for Mersenne is found at the beginning of Lesson 8. Marin Mersenne was a 17^{th} century monk who studied the numbers $2^n - 1$. These can only be prime if n is prime, but that is no guarantee of primality as seen in the homework.

Euclid showed the known perfect numbers were of the form $2^{p-1} \times (2^p - 1)$. Euler proved even perfect numbers could only be in this form. It remains an open question whether or not there are any odd perfect numbers. Another perfect number is generated, whenever a Mersenne prime is found. The 47^{th} Mersenne primes was reported April 12, 2009. The exponent is $n = 42643801$. The largest known prime is usually a Mersenne prime. GIMPS[2] involves the author and some students in this search.

Prime numbers have been used extensively in cryptology used to hide messages. Some numbers have become restricted or illegal to possess, utter, or propagate by the general public, such as those used to encode music and videos on DVDs.[3]

In addition to the search for perfect numbers, the GIMPS project also helps in finding small factors for Mersenne numbers using the Elliptical Curve Method (ECM).

[2] `http://www.utm.edu/research/primes/mersenne.shtml`
[3] See: `http://en.wikipedia.org/wiki/Illegal_number`

4.9 Prime Homework

Each problem is worth three points.

1. What is the sum of the proper divisors of $2^4 \times (2^5 - 1)$ and $2^6 \times (2^7 - 1)$? Are these numbers perfect?

2. For the number 220, find all the factors; add the factors (except itself); count all the factors; find the prime factorization.

3. For the number 284, find all the factors; add the factors (except itself); count all the factors; find the prime factorization.

4. Extend the Sieve of Eratosthenes to find the prime numbers between 101 and 200. **Bonus points** for defining and identifying any **prime decades.**

101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130
131	132	133	134	135	136	137	138	139	140
141	142	143	144	145	146	147	148	149	150
151	152	153	154	155	156	157	158	159	160
161	162	163	164	165	166	167	168	169	170
171	172	173	174	175	176	177	178	179	180
181	182	183	184	185	186	187	188	189	190
191	192	193	194	195	196	197	198	199	200

5. How large a factorial can you calculate **exactly** using a **TI-84** calculator? A **TI-$n$spire** calculator?

6. Add the first few odd numbers together. Initially, just the first one. Write it down as sequence member number one. Then add the first and second $(1 + 3)$. Write it down as sequence member number two. Then the first three, *etc.* until you have added the first five together. Symbollically this can be expressed as: $\sum_{i=0}^{n} 2i+1$ for $n \in \{0, 1, 2, 3, 4\}$. What pattern is there in the resultant sequence?

7. Prime factor 2047 otherwise known as $2^{11} - 1$.

8. For both parts, write out the prime factorization of the original numbers. **Bonus points** for Venn diagrams!
   (a) Find the GCF(156,182).
   (b) Find the LCM(496,8128).

9. Find the least common multiple and the greatest common factor of:
   a) 60, 72 b) 12, 20, 36 c) 9, 12, 14

10. Prime factor 1001.

11. **bonus:** Bob has every sixth night off from work. It happens that tonight has his favorite shows that only come on once a week and he is off to watch them. How long until he gets to watch his shows again?

Numbers Lesson 5

Powers, Bases/Conversion, Pascal's Triangle

The taxicab number of $1729 = 7 \times 13 \times 19$ was dull.
1729 is a very interesting number. It is the smallest integer which is the sum of two cubes multiple different ways.

Paraphrase of Hardy and Ramanujan

In this lesson we will examine ways to express the natural numbers, bases, powers, and some other important catagories of natural numbers. We will also explore some related topics such as parity, Fermat numbers, and Pascal's triangle.

5.1 The Prince of Amateur Mathematician: Fermat

Pierre de Fermat was an amateur mathematician living in the early 1600's (1601–1665) who had a profound influence on mathematics for the last four centuries. By amateur we mean Fermat earned his living by doing other work and mathematics was purely a hobby. Fermat was a jurist, which means he had a law degree and practiced law. In his job he was supposed to avoid social contact and this probably gave him more time to devote to mathematics. With Pascal he developed the theory of probability and independent of Descartes he developed analytic geometry. He also developed many important concepts which led into the development of calculus. In this lesson we will explore the numbers which were named after him.

Perhaps Fermat's most famous legacy is known as Fermat's Last Theorem. After Fermat died his son found written (about 1637) in the margin of his textbook by Diophantus the equation $x^n + y^n \neq z^n$, where $n > 2$ along with the statement: "I have discovered a truly marvelous proof of this, which, however, the margin is not large enough to contain." This is a generalization of the Pythagorean Theorem (where $n = 2$). This became known as Fermat's Last Theorem (now FLT) because it remained after all his other theorems had been solved. The theorem part of the name was also a misnomer until it was actually proved in 1993/4. More on both theorems is in Lesson 12.

There is also an important theorem known as Fermat's Little Theorem which forms the basis of some primality testing: If p is a prime number, then for any integer a, $a^p - a$ is evenly divisible by p ($a^p \equiv a \pmod{p}$).

5.2 Powers, Exponents, Base 10

The expression x^n is called a **power** where, n is the **exponent** and x is the **base**.

Example: $2^{10} = 1024$, 1024 is a power of 2, specifically it is 2 multiplied by itself 10 times: $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$. **Exponentiation** is a shorthand notation for such repeated multiplication.

Most people have 5 digits ("fingers") on each hand and 2 hands. This has led to the use of the decimal system of notation with 10 digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. We express our numbers using **place value** where each position to the left is weighted 10 times the position to its right. Thus $1331 = 1 \times 10^3 + 3 \times 10^2 + 3 \times 10^1 + 1 \times 10^0 = 1000 + 300 + 30 + 1$. This system of writing numbers is the **Hindu-Arabic Number System** or Arabic Numerals.

5.3 Roman Numerals

We already encountered in the homework for Numbers Lesson 1 the Roman Numeral System. We wish to formalize here some information about them and make certain you are familiar with them.

The following symbols have the following values: I=1; V=5; X=10; L=50; C=100; D=500; and M=1000. Lower case can also be used, especially for small values: i=1; v=5; x=10; l=50. Smaller values go to the right unless they represent subtraction. The restrictions for subtraction are: 1) you can subtract no more than one symbol; 2) that symbol can not be more than an order of magnitude less; and 3) it must also be a power of ten. Thus 49 = XLIX, but not IL and 45 = XLV, but not VL.

5.4 Properties of Exponents

"Anything" to the zero power is 1: $x^0 = 1$ (x cannot equal 0.)

Anything to the first power is itself: $x^1 = x$.

Properties of Exponents:

1. Product of two powers with like bases: $x^a \times x^b = x^{a+b}$.
2. Quotient of two powers with like bases: $x^a/x^b = x^{a-b}$.
3. Power of a power: $(x^a)^b = x^{ab}$.
4. Power of a product: $(xy)^a = x^a y^a$.
5. Power of a quotient: $(x/y)^a = x^a/y^a$.

Notice how the place value system was not possible before zero was invented (some insist discovered!).

One **order of magnitude** means one power of ten.

A $Ke^{i\theta}$ term is **order of bagnitude,** or binary order of magnitude, which means one power of two.

Some powers have special names like x^n where $n = 2$ are called **squares** and for $n = 3$ are called **cubes**. Some times the term perfect square or prefect cube is used not in the sense of perfect number but in the sense of being the square of a rational number, like $2^2 = 4$ and not the square of an irrational number, like $\sqrt{5} \cdot \sqrt{5} = 5$. Five is not considered a "perfect square."

5.5 Base 11, Base 12, Converting from Base 10

The number above (1331) could just as easily be expressed in base 11 as $1000_{11} = 1 \times 11^3 + 0 \times 11^2 + 0 \times 11^1 + 0 \times 11^0$. Note: when no base is indicated (usually via a subscript afterwards), base 10 is assumed. Maybe you prefer base 12, where $92E_{12} = 9 \times 12^2 + 2 \times 12^1 + E \times 12^0$, and T represents the digit "ten" and E represents "eleven" in our **duodecimal** system. The following example also illustrates how to convert from base 10 to another base by repeated division and use of the remainders.

$$\begin{array}{r l}
9 & \text{R } 2 \\
12\overline{)\ 110} & \text{R } 11 \text{ or "E"} \\
12\overline{)1331} &
\end{array}$$

5.6 Base 2, Base 4, Base 8, and Base 16; Converting to Base 10

The computer revolution has expanded the use of bases 2, 8, and 16 especially. A typical base 2 number might be (the character "6" in EBCDIC):

$$\begin{aligned} 11110110_2 &= 1\times 2^7+1\times 2^6+1\times 2^5+1\times 2^4+0\times 2^3+1\times 2^2+1\times 2^1+0\times 2^0 \\ &= 128+64+32+16+0+4+2+0 \\ &= 246 \end{aligned}$$

Base 2 is also called **binary**. Base 8 is known as **octal**. **Hexadecimal**, or affectionately called **hex** for short, refers to base 16.

Since 4, 8, and 16 are powers of 2, it is an easy matter to convert such a number from base 2 to base 2^n. You regroup bits n at a time from the right. For example: $11110110_2 = 3312_4 = 366_8 = F6_{16}$.

In base 16, we need names for our 6 additional "fingers" (I mean digits). The usual choices are A, B, C, D, E, and F. Below is a table of how the numbers are represented in the common bases.

Each binary digit is called a **bit**.

Each hexadecimal digit (or 4 bits) $Ke^{i\theta}$ calls a **hit** (hex digit).
It is more commonly called a **nibble**.

8 bits make a modern **byte**.[1] (Hence the term nibble above for half a byte.)

Among the many definitions of bit is another important historic and mathematical meaning. The US dollar originated out of the Spanish-American peso or piece of eight, which could be broken into eight parts called bits. Hence 2 bits is the equivalent of a modern US quarter and 8 bits is a dollar.

Note how close in magnitude $10^3 = 1000$ and $2^{10} = 1024$ are.

The term **kilo** (see Numbers Lesson 10) which really is 10^3 now often means 2^{10} (1024).
The term **mega** which really is 10^6 now often means 2^{20} (1,048,576).
The term **giga** which really is 10^9 now often means 2^{30} (1,073,741,824).
The term **tera** which really is 10^{12} now often means 2^{40} (1,099,511,627,776).

[1] Historically a byte ranged from 6 to 12 bits.

Base 16	Base 10	Base 2
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

For a good demonstration of adding binary numbers see the video at `http://www.woodgears.ca/marbleadd/index.html`

5.7 Parity

Parity is a term now commonly used in computer storage and communications. The word is related to par as in golf where "he hit under par" and connotes equivalence. In computers, it relates to base 2 and there are several types: even, odd, mark, and no. Even parity typically means a bit will be appended to each byte (or word) to force an even number of bits. For example, the character "1" in the ASCII communication code is 31_{16} or 00110001_2. If transmitted or stored with even partiy, this byte would have an additional bit appended and that bit would be set (=1) for there to be an even number of bits set. Odd parity would mean the appended bit would be reset (=0). Errors can then be detected if the received or recalled value does not have the correct parity. More advanced encoding schemes (LRC, CRC, Hamming, *etc.*) allow error correction as well, but require additional storage. Mark indicated the parity bit is always set (=1). No parity indicates the parity bit is either not present or equal to zero.

5.8 Other Bases

An interesting application of base 3, known as **ternary**, can be read about in an article in the *American Scientist*, July–Aug. 1998, pg 314–9. There is no reason the base has to be positive. A homework problem will deal with base -3. Base 60 was developed by the ancient Babylonians. We still use it for time (60 seconds = 1 minute; 60 minutes = 1 hour) and angle (60 seconds = 1 minute, 60 minutes = 1 degree; $6 \times 60 = 360$ degree = 1 circle) measurements. A fun base can be base 26 and will also be dealt with in the homework. The letters of the English alphabet are an obvious choice for "digits."

5.9 Fermat Numbers

Fermat noted that $2^{2^0}+1 = 2^1+1 = 3 = F_0$ was prime as was $2^{2^1}+1 = 5 = F_1$, $2^{2^2}+1 = 2^4+1 = 17 = F_2$, $2^{2^3}+1 = 2^8+1 = 257 = F_3$, and $2^{2^4}+1 = 2^{16}+1 = 65537 = F_4$. He conjectured that $2^{2^n}+1 = F_n$ was always prime. In 1732, Leonard Euler, another famous mathematician, showed that $2^{2^5}+1 = 2^{32}+1 = F_5 = 4294967297$ was divisible by 641. The search for prime factors of larger Fermat numbers continues and is another potential EXPO Project.

In 1796, Gauss used Fermat numbers in his proof that a regular heptagon (7-sided polygon) was not constructible, whereas the regular heptadecagon (17-sided polygon) was. Please note that F_n usually refers to Fermat numbers which is why we used L_n for Fibonacci numbers in Numbers Lesson 2. (**Note also:** Most calculators process stacked exponents left to right and not right to left as mathematicians would expect, thus parentheses are highly recommended.) Before the 1977 Fortran standard Fortran compilers were notoriously schizophrenic on how this was interpretted. The TI-82/3/4 series of calculators still is, with a different order used depending on whether the $\wedge$ or $^{-1}$ symbols is used! (Compare $3\wedge3\wedge(-1)$ with $3\wedge3^{-1}$.)

5.10 Pascal, Pascal's Triangle

Blaise Pascal was yet another famous mathematician contemporary with Fermat with whom he shares the honor of inventing probability. His biography is located in Section 10.1. Pascal's Triangle is useful in many diverse fields of mathematics and is displayed below:

$$\begin{array}{ccccccccccccl}
 & & & & & 1 & & & & & & & \text{row } 0 \\
 & & & & 1 & & 1 & & & & & & \text{row } 1 \\
 & & & 1 & & 2 & & 1 & & & & & \text{row } 2 \\
 & & 1 & & 3 & & 3 & & 1 & & & & \text{row } 3 \\
 & 1 & & 4 & & 6 & & 4 & & 1 & & & \text{row } 4 \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \text{row } 5 \\
\iddots & & \ldots\ldots\ldots & & & \vdots & & & \ldots\ldots\ldots & & \ddots & & \text{row } n
\end{array}$$

Notice how each entry is the sum of the numbers diagonally above it to the left and to the right—where missing numbers on the sides can be assumed to be zero. Notice how we already saw the first few rows in the homework as the powers of 11! Each entry in Pascal's triangle can also be found as: ${}_nC_r = \frac{n!}{r!(n-r)!}$, where n is the row number and r goes from 0 to n for each position in the row. An alternate notation for these binomial coefficients is:

$$\boxed{\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1} \text{ or } {}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}.}$$

Pascal's Triangle was well known to the Chinese 300 years before Pascal where it was used to extract n^{th} roots. However, Pascal was the first to apply it to games of chance between two people.

Example: The recursive definition of factorials is useful for simplifying combinations or ${}_nC_r$. ${}_9C_6 = \frac{9!}{6!3!} = \frac{9\cdot 8\cdot 7\cdot 6!}{6!\cdot 3!} = \frac{9\cdot 8\cdot 7}{3\cdot 2}$, where we have explicitly show the recursive definition of factorial and the common factor of 6! has been cancelled.

5.11 Golden Rule, FOIL/Box

Although we will formally define binomial in Numbers Lesson 13, a quick review of algebra will be included here. First, when dealing with equations, it is important to always follow the "golden rule:" "what you do to one side, always do to the other." This is partially formalized as two axioms as follows:

Additive Property of Equality: If $a = b$, then $a + c = b + c$.
Multipicative Property of Equality: If $a = b$, then $ac = bc$.

Also, notice what happens when 23 is multiplied by 12:

$$\begin{aligned}
(12)(23) &= (10\cdot 20) + (10\cdot 3) + (2\cdot 20) + (2\cdot 3) \\
&= 200 + 30 + 40 + 6 \\
&= 276
\end{aligned}$$

$$\begin{array}{rr} & 23 \\ \times & 12 \\ \hline & 46 \\ + & 23 \\ \hline & 276 \end{array} \qquad \begin{array}{rr} & 12 \\ \times & 23 \\ \hline & 36 \\ + & 24 \\ \hline & 276 \end{array}$$

This is an important algorithm to remember when multiplying binomials such as $(x+1)(x+1) = x^2 + x + x + 1 = x^2 + 2x + 1$, and is often referred to as the FOIL method, an **acronym** for **F**irst, **O**uter, **I**nner, **L**ast. However, the **box method** generalizes to higher order polynomials.

	$2x$	$-3y$
x	$2x^2$	$-3xy$
$-2y$	$-4xy$	$6y^2$

So, $(2x-3y)(x-2y) = 2x^2 - 7xy + 6y^2$.

5.12 Binomial Theorem or Formula

The Binomial Theorem or Formula using Pascal's Triangle can be useful for evaluating binomials raised to powers:

$$\boxed{(x+y)^n =_n C_0 x^n y^0 +_n C_1 x^{n-1} y^1 +_n C_2 x^{n-2} y^2 + \ldots +_n C_n x^0 y^n}$$

Example:

$(3x+4)^5 = 1\cdot(3x)^5(4)^0 + 5\cdot(3x)^4(4)^1 + 10\cdot(3x)^3(4)^2 + 10\cdot(3x)^2(4)^3 + 5\cdot(3x)^1(4)^4 + 1\cdot(3x)^0(4)^5$

Of course, $4^0 = (3x)^0 = 1$, $(3x)^1 = 3x$, and $4^1 = 4$ so this might be written as follows before simplifing further.

$(3x+4)^5 = 1\cdot(3x)^5 + 20\cdot(3x)^4 + 10\cdot(3x)^3(4)^2 + 10\cdot(3x)^2(4)^3 + 5\cdot(3x)(4)^4 + 1\cdot(4)^5$

However, it has now lost the obviousness of the pattern, where each coefficient comes from a line in Pascal's Triangle, one set of exponents are decreasing, while the other set is increasing. For any term, the exponents sum to the power, in this case 5.

5.13 Base Homework

Each problem is worth two points.

1. Write out the definition of **googol** from a good dictionary.

2. Write out the definition of **googolplex** from a good dictionary.

3. Compare the American, French, British, and German number systems for the term billion and milliard. (See next problem.)

4. When we think of large numbers, we think of thousands, millions, billions, and trillions. Find a good dictionary that extends the concept of numbers beyond trillion and write a few down.

5. Solve for x and z: $2^2 \times 2^3 = 2^x$ and $(2^2)^3 = 2^z$.

6. Clearly apply the FOIL method to expand $(x + 1) \times (x + 1)$.

7. Write a huge number using ONLY three 9's (and nothing else).

8. Use Pascal's Triangle to expand $(x + 1)^3$.

9. Calculate 2^{20} and 2^{30}. Compare (the relative or percent difference) with 10^6 and 10^9, respectively.

10. Knowing that the number of dominoes in a set is a **triangular number**, and that there are 28 dominoes in a double 6 set, calculate the number of dominoes in a double 9, double 12, double 15, and double 16 set.

11. Madam I'm Adam. Name no one man. Some numbers are **palindromes.** (Look it up in dictionary). Write at least five of the fifteen, **prime,** three digit palindromes. (Use Homework 3, problem 4 for reference.)

12. Convert 1101_2 into base 10.

13. Convert 27 into base 2.

14. Consider \$1.17 as 117 pennies and convert it into the smallest number of quarters, nickels, and pennies. Write this as a base 5 number of pennies.

15. Convert 234_5 (2 quarters, 3 nickels, 4 pennies) into a base 10 number of pennies.

16. Change 38 days into weeks and days.

17. Change 210 hours into days and hours.

18. Change \$2.69 into the smallest number of coins consisting of quarters, dimes, nickels, and pennies.

19. Change $A3B5_{16}$ into base 10.

20. Multiply and simplify: a) $(2x-5)(3x+2)$ b) $(x+3)(x-7)$.

21. **Bonus:** How many what is a crore? What is its value in US dollars?

22. **Bonus:** How much modern American change can you have and not be able to make change for a dollar?

Numbers Lesson 6

This is a Lie!

Cogito ergo sum.[1] René Descartes

In this lesson we give an overview of the field of logic. We introduce if-then statements, logical shorthand, negation, converses, inverses, and the contrapositive. Deductive and inductive reasoning are introduced along with direct and indirect proof. First we have to think our way into existence.

6.1 Father of Modern Mathematics: René Descartes

René Descartes, the early French mathematician (1591–1650) spent considerable time philosophizing about mathematics and its very existence. To get started he had to assume his very own existence in his famous quote (in Latin): "*Cogito ergo sum,*" which means, "I think, therefore I am."

Descartes studied law but never practiced it, choosing instead to travel Europe as a mercenary soldier. In this way he met lots of people and had many useful experiences. There is speculation he acted as a spy in this way. When Galileo was condemned by the Catholic Church, Descartes abandoned plans to publish a great work he had written.

Descartes was also a key figure in the scientific revolution. He invented analytic geometry with the cartesian coordinate system which is named after the latinized version of his name. This invention revolutionized mathematics by forming a strong connection between geometry and algebra. Descartes also spent considerable time in bed, rarely getting up before noon. It has been said he developed the cartesian coordinate system while lying in bed watching a fly on the ceiling and trying to describe its movements. Descartes also created exponential notation, the use of superscripts to indicate repeated multiplication.

[1] I think, therefore I am.

Descartes died in Stockholm, Sweden while tutoring the queen there. Although he died of pneumonia, the fact of having to get up early and ride across town to tutor the queen in that cold environment is said to have been the major cause.

6.2 Hypothesis, Conclusion, Conjecture

A **premise** (also known as an **antecedent** or **hypothesis**) is a tentative assumption made in order to draw out and test its logical or empirical consequences.

A **consequence** or **conclusion** is the necessary result of two or more propositions taken as premises.

Sentential logic or propositional logic, consists of a sentential language, a semantic interpretation of that language, and a sentential derivation system. Predicate logic goes further and builds on sentential logic. We give here the merest overview of this broad field.

6.3 Deductive *vs.* Inductive Reasoning

As stated in the first two lessons, Geometry often deals with proofs. Proofs are based on logical reasoning which follow two basic types.

Deductive (or logical) Reasoning is the process of demonstrating that if certain statements are accepted as true, then other statements can be shown to follow from them.

Inductive Reasoning is the process of observing data, recognizing patterns, and making generalizations from the observations.

Both are important to mathematics in general and to Geometry specifically.

The generalization used in inductive reasoning is called a **conjecture.**

A **statement** is a declarative sentence which is either true or false, but not both. **Proposition** is often used interchangely with the term statement. A **paradox** is a sentence which is both true and false, such as "I am lying" (*cf* Titus 1:12). A **simple statement** is a statement containing no connecting words. **Compound** or **complex statements** are formed from simple statements using basic connection. The basic connections are: and, or, if... then..., if and only if, not. Often, other connecting words such as unless, because, either/or, neither/nor, although, nevertheless, except, but (save), only, as, since, *etc.* are used which can be restated using the basic ones.

Examples:

"*Unless* he is careful, he will crash." means the same as "*If* he is *not* careful, then he

will crash."
"*Whenever* I tell a joke, my students laugh." is equivalent to "*If* I tell a joke, *then* my students laugh." except for some circumstance of time.

This definition of statement is based on an axiom of Aristotle (ancient Greek philosoper (*c.* 384–322 B.C.)) called the **law of excluded middle**. Symbollically, $p \vee \overline{p}$. (This is very similar to the **principle of bivalence** which states every proposition is either true or false, but not equivalent! There are logics with one and not the other.) If we reject this axiom, **fuzzy logic** involving probability is the result. In recent years, fuzzy logic has started to invade your cars and homes (washing machines, *etc.*), and is "the rage."

When translating declarative statements into logical form it is common to recast things in the present tense. This assumes that time relationships are not important to the argument. As noted above, whenever certain common words are, used the sentence should be recast using the standard if-then syntax of logic.

The following statements may be equivalent and useful for this task:

- If apples are on sale, then I buy apples.
- Whenever apples are on sale, I buy apples.
- Because apples are on sale, I buy apples.
- I buy apples since they are on sale.
- I buy apples unless apples are not on sale.
- I buy apples except when apples are not on sale.
- I buy apples save when they are not on sale.
- I buy apples as they are on sale.
- I buy apples until they are not on sale.

The following statements may be somehow different and you might try your hand at recasting them in standard form.

- I buy apples only if they are on sale.
- Although apples are not on sale, I buy apples.
- I buy apples whether or not they are on sale.
- I buy apples either if they are on sale or if they are not on sale.
- I buy apples neither when they are on sale, nor if they are not on sale.
- Apples are not on sale, nevertheless I buy apples.

6.4 Logical Shorthand

Short hand notation is often used when writing logical arguments. Statements such as "I have a job." may be replaced by p and the **conditional** statement, "If I have a job, then I must work." might be replaced by $p \rightarrow q$, where q in this case is equivalent to "I must work." A conditional is also known as an **implication**. An if-then statement can be rewritten using the word implies, and in fact, the symbol $\rightarrow$ is often read that way. Some reasoning is **valid**, in that it gives correct or truthful results whereas some is **faulty** or **invalid**. You may think the old adage: "Watch your p's and q's" is derived from the extensive usage of these symbols. However, it actually is drinking advice to watch ones **p**ints and **q**uarts!

A **theorem** is a statement that has been proven, or can be proven, from the postulates.

A **corollary** is a result which follows naturally, or a specific application of a theorem. A **lemma** is a mathematical statement proven not for its own sake, but for use in proving a more important statement called a theorem.

Modus Ponens (MP) says that if $p \rightarrow q$ is true and p is true, then q must be true. This principle is also known as the **Law of Detachment** (LD).

Modus Tollens (MT) says that if $p \rightarrow q$ is true and q is false (not true), then p must be false. MT is essentially equivalent to the **Law of indirect Reasoning** (below) and is the basis for proof by contradiction.

Example: consider the following conditional statement: If the weather is beautiful, then we'll go for a walk. MP implies that if p is true (The weather is beautiful.) q is also true (We'll go for a walk.). MT implies that if $p \rightarrow q$ is true (If the weather is beautiful, then we'll go for a walk.) and q is false (It is not the case that we'll go for a walk.) then p is false (The weather is not beautiful.).

It is a good thing when a system of axioms is consistent, sound, and complete.

Consistent means none of the theorems contradict one another.

Soundness means the system's rules of proof will never allow a false inference from a true statement.

Complete means all true statements can be proved within the system.

Unfortunately, no useful system of arithmetic can be both consistent and complete (Gödel's Incompleteness Theorem).

6.5 Negation/Double Negation, Converse, *etc.*

The **negation**, symbollically $\sim p$, $\overline{p}$, or -p, of a statement is very useful. If p is "I have a job," then $\sim p$ is "I do not have a job."

The **double negation**, as taught in English (not Spanish!) gives back the original statement! $\sim(\sim p)$ is equivalent to p. If it is not true, that "I do not have a job." Then it must be true "I have a job."

The **Converse** of $p \to q$ is $q \to p$.

The **Inverse** of $p \to q$ is $\sim p \to\sim q$.

The **Contrapositive** of $p \to q$ is $\sim q \to\sim p$.

Law of Contrapositive (LC) states that if a conditional is true, so is its contrapositive.

Continuing the weather example above, the contrapositive would be "If we'll not go for a walk, then the weather is not beautiful." LC tells us this is true if the original statement is true. It should be easy to see that the **converse of the inverse is the contrapositive.**

Whether the conditional is true does not affect whether the converse is true.

A **counterexample** is an example of a conditional statement being false.

Sometimes, instead of writing a long proof to determine something is true, many will try to find a counterexample.

An "if and only if" (often abbreviated **iff**) statement is called a **biconditional** and combines the statements $p \to q$ and $q \to p$ into $p \leftrightarrow q$. To prove a biconditional, one proves the corresponding two conditionals.

A **syllogism** is composed of a **major premise**, a **minor premise**, and the resulting conclusion.

A syllogism has three parts. Therefore, this is not a syllogism. (ha ha ha).

The consequence is often preceeded by the word **therefore** which is also often abbreviated by three dots arranged in a triangle pointing up ($\therefore$).

The **Law of Syllogism** is also called the **Law of Transitivity** (see also Numbers Lesson 14) and states: if $p \to q$ and $q \to r$ are both true, then $p \to r$ is true.

Reasoning and also definitions are sometimes said to be **circular.**

6.6 Direct *vs.* Indirect Proof

Mathematical proofs come in two basic flavors known as direct and indirect. We already saw an example of an indirect proof in Numbers Lesson 4 when we proved by contradiction that primes form an infinite set. Proof by contradiction is also known as using the law of indirect reasoning.

Law of Indirect Reasoning:
If valid reasoning from a statement p leads to a false conclusion, then p is false.

Any proof using the Law of Contrapositive (above) or the Law of Ruling out Possibilities (below) are also classified as indirect proofs.

Law of Ruling out Possibilities:
When statement p or statement q is true, and q is not true, then p is true.

We will see further examples of these five laws of logic in Chapter 11 of our Geometry textbook and my associated supplement.

6.7 Model Theory and Mathematical Models

Traditionally logic was a part of philosophy and one of the three subjects studied together: grammer, logic, and rhetoric. Since the mid-1800's it has been studied as a part of the foundations of mathematics. It is important for a full understanding of fallacies and paradoxes. Set theory has largely replaced the role of logic in the development of mathematics.

There are variations on logic and extensive discussions which link logic with various schools of philosophy. The field has changed extensively within the last 100 years with the development of first order logic. First order logic extends propositional logic by allowing quantification over individuals in a universe of discourse. The symbols used are: $\forall$ meaning **"for all"** and $\exists$ meaning **"there exists."** Second order logic allows quantification over sets. Second order logic is required for full use of real numbers (least upper bound).

We can combine axioms with a logic system to develop model theory. This use of the word model in mathematics is different and more recent than the mathematical models one might construct to describe some scientific phenomenon.

6.8 Logic Homework

Each problem is worth two points, except as noted.

Given p = "This is a frog.", q = "It should croak." Write out **in words** the following:

1. Conditional ($p \rightarrow q$).

2. Inverse ($\sim p \rightarrow \sim q$).

3. Converse ($q \rightarrow p$).

4. Contrapositive ($\sim q \rightarrow \sim p$).

5. Biconditional ($p \leftrightarrow q$).

6. (6 points) Write out **in words** the indicated conditional statements for the following sentence: "If I get my allowance today, I'm going to buy my favorite DVD."

   (a) Inverse:

   (b) Contrapositive:

   (c) What can you conclude if you are told, "I bought my favorite DVD."?

7. Given a compound statement: My sister, who cooks whenever she can, loves cooking for people as long as they are appreciative of her labors. Write this statement in shorthand, symbolically identifying each piece.

8. Give a counterexample of: "Bears are large and dangerous to approach."

9. Lots of advertising tries to appeal to a human need to belong. Write one counterexample for each of the following suggestive advertisements. "You'll be cool if you buy *Converse* shoes." "Buy a *Lexus* automobile, then everyone will be dripping with envy."

10. Rewrite the sentence as conditional statement: All squares are rhombi.

11. Write the converse and state if it is true: If you are a driver, then you are at least 16.

12. Form the converse to: "We'll go to the fair if they announce square-dancing over the radio."

13. "If you go fishing, you are sure to hook a trout." You bring home a trout for supper. Did you catch it? Explain your answer.

14. Given: "If a golfer has won the U.S. Open Tournament, then [s]he is in the major leagues." What can you conclude about these two people? Tiger Woods won the U.S. Open Tournament. Bernhard Langer has not won a U.S. Open Tournament.

15. What can be concluded from: "If a nail is lost, then a shoe is lost. If a shoe is lost, then a horse is lost. If a horse is lost, then a rider is lost. If a rider is lost, then a battle is lost. If a battle is lost, then a kingdom is lost."?

16. See Section 2.2 of your geometry textbook for further examples. Several problems from prior editions were assigned in the past.

17. Base 26 can be fun. Convert your first name/nickname from base 26 into base 10. Try to restrict your first name to 6 letters to avoid 32-bit integer overflow. Let $A = 1, B = 2, ..., Y = 25, Z = 0$, ignore upper/lower case.

18. **Bonus:** Express the numbers 8 through 12 in base -3. Use 0, 1, and -1 as your digits. Check out the article Third Base in the Nov./Dec. 2001 issue of American Scientist.

Numbers Lesson 7

To Tell the Truth

A theorem a day Means promotion and pay!
A theorem a year And you're out on your ear! Paul Erdös

We have already seen in Numbers Lesson 1 the relationship between union (disjunction) and **or** as well as intersection (conjunction) and **and**. Here we will also introduce various symbols used when drawing logic diagrams, give truth tables in two different forms for a few other common operator, and explore how *and* and *or* are similar to switches in series and parallel circuits.

Exclusive or is discussed along with DeMorgan's Law, tautology, and contradiction. We close after touching on nands, nors, flip-flops, and logic equations. First we discuss a mathematician world-reknown for his logical development of Geometry.

7.1 The Father of Geometry: Euclid

Euclid of Alexanderia was an important Greek mathematician living around 300 B.C., his exact lifespan is unknown. Euclid was born in Greece but spent much of his life near the great library in Alexandria, Egypt.

Euclid wrote the 13 volume series of books known collectively as the *Elements*. It became the most successful mathematical textbook ever. Several online versions[1] of the *Elements* exist, including a wonderful color version[2] from the early 1800's. In the *Elements* Euclid assumes five axioms and develops the whole of euclidean geometry from them. Euclid's fifth postulate (or variations thereof such as "through a point outside a line one and only one line can be drawn parallel to the given line.") became very controversial by the early 1800's. In addition to geometry, many number theory ideas are explored and proven in the *Elements*. These include the form of even perfect number and the infinitude of primes, An algorithm to find the greatest common factor also bears his name. Euclid summarized much of the known mathematics of his time.

[1] http://aleph0.clarku.edu/~djoyce/java/elements/elements.html

[2] http://www.math.ubc.ca/people/faculty/cass/Euclid/byrne.html

Figure 7.1: $2 \times n$ Truth Tables for And, Or, Eor, and Not.

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

eor	0	1
0	0	1
1	1	0

p	$\overline{p}$
0	1
1	0

7.2 Truth Tables

Numbers Lesson 6 introduced the concept of logical **statements** and **connectives** used to joined them into **compound** statements called **arguments**. Here we explore the **conclusion** of these arguments as the input statements take on various values of **true** or **false**. Since the connectives we are studying (and, or, if-then, iff) and negation (not) are **truth-functional** (its truth value can be figured out solely on the basis of its components), we can evaluate these arguments by exhaustively listing all possible values these inputs may take on. If there are n components, there will be 2^n rows in the corresponding truth table.

Parentheses should be used when combining multiple compound statements together with connectives. If parentheses are omitted, the following order of operation should [generally] be assumed: biconditional (highest), conditional, conjunction/disjunction, and negation (lowest).

Given in Figure 7.1 are truth tables in the form of multiplication and addition tables. You might compare these with those found in the homework for lesson 2 for multiplying and adding even numbers.

7.3 Ands, Ors, Exclusive Ors

Since the **and** and **or** tables above are so similar to the multiplication and addition tables seen earlier, **and** is often symbolized by $\bullet$ or $\wedge$ (similar to intersection) and **or** is often symbolized by $+$ or $\vee$ (similar to union). $|$ is also often used for *or*. Be <u>very</u> careful when programming since conventions vary widely between programming language! Languages such as C and C++ introduce additional confusion by differentiating between bitwise (operating on each bit in a string) and logical operators (only treating the value as zero or not zero).

Augustus De Morgan's (1806–1871) major contribution to mathematics was reforming logic and establishing symbolism for algebra. He was the one to define and introduce mathematical induction, which up to that point was still unclear. One ma-

Figure 7.2: Contingency Table for Two Variables and Many Operators.

p	q	$\overline{p}$	$\overline{q}$	$p \wedge q$	$p \mid q$	p eor q	$p \rightarrow q$	$p \leftrightarrow q$	$\overline{p \wedge q}$	$\overline{p \vee q}$	$\overline{p \text{ eor } q}$	$p \wedge \overline{p}$	$p \vee \overline{p}$
0	0	1	1	0	0	0	1	1	1	1	1	0	1
0	1	1	0	0	1	1	1	0	1	0	0	0	1
1	0	0	1	0	1	1	0	0	1	0	0	0	1
1	1	0	0	1	1	0	1	1	0	0	1	0	1

jor result known as **De Morgan's Law** is summarized below in two different formats.

De Morgan's Law: $(A \cap B)' = A' \cup B'$ and $(A \cup B)' = A' \cap B'$.

De Morgan's Law: $\overline{A \wedge B} = \overline{A} \vee \overline{B}$ and $\overline{A \vee B} = \overline{A} \wedge \overline{B}$.

The major author debugged a significant number of COBOL programs by checking logic of this form.

Around the same time, George Boole (1815–1864) was also establishing logic symbolism. Boolean Algebra, which is a foundation for computers, is an algebra of sets with the operators of union and intersection. Equivalently it is an algebra with the numbers 0 and 1 and operators of **and** and **or**. More details are available in Numbers Lesson 14.

As noted above, truth tables appear in two basic forms: 1) as multiplication or addition style tables; and 2) as an exhaustive list of possible values. Take a moment and compare these truth tables with those obtained in the homework in Numbers Section 2.11 regarding the addition and multiplication of even numbers. Then, compare the format used in Figure 7.2 with the format used in Figure 7.1 in this lesson.

Often in a truth table the symbol **T** for **true** is used for 1 and the symbol **F** for **false** is used for 0.

Note how *neor* and the biconditional are the same.

Another common name for the **biconditional** is **equivalence**.

If a proposition contains only **1**'s (**T**'s) in the last column of its truth table, it is a **tautology**. (See $p \vee \overline{p}$ in the table above.)

If a proposition contains only **0**'s (**F**'s) in the last column of its truth table, it is a **contradiction**. (See p **and** $\sim p$ in the table above.)

An argument is **valid** if it has good logical structure, otherwise it is **invalid**. An argument is **sound** if and only if it is valid and has true premises, otherwise it is

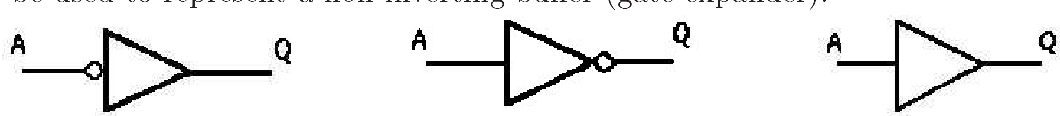
unsound. One also uses the following terms to identify statement requirements in A implies B: **necessary**: (B cannot be true unless A is true, or **sufficient**: A cannot be true unless B is true. A **fallacy** uses a false premise, invalid reasoning, or vague/ambiguous language. One also calls a set of statements either **inconsistent** if they lead to a contradiction or **consistent** if not. A set of statements is **complete** if one can determine for any combination of statements a result (*i.e.* prove it) or else **incomplete.** Kurt Gödel, whose biography appears in Sec. 14.1, in 1931 showed that no complete system that admits the natural numbers (Peano axioms) can be consistent, which is now known as **Gödel Incompleteness Theorem.** Thus any useful logical system must either be inconsistent or incomplete. This derailed attempts to axiomize all of mathematics.

If a proposition contains both **1**'s and **0**'s (**T**'s and **F**'s) in the last column of its truth table, it is a **contingency**.

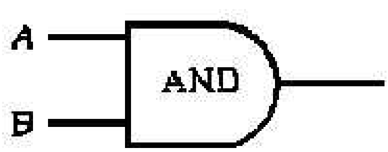
Forming truth tables like this is a common way to compare the validity of two different statements. Actually, few of the 16 possible combinations of 0's and 1's are missing in the table above. A former teacher of the major author added a **let** operator to complete the list—those were his initials!

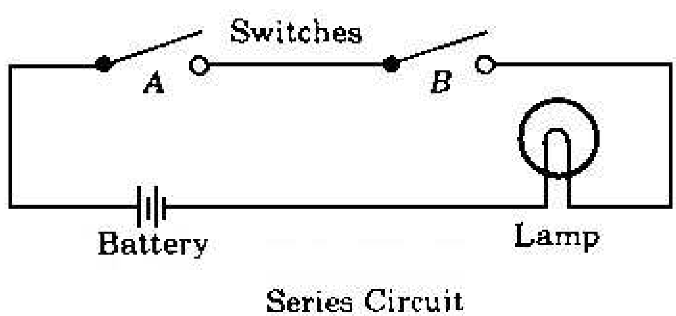
7.4 Logical Symbols

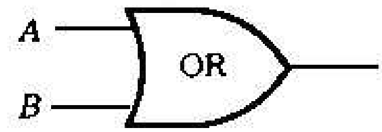
Given below are three symbols commonly used to represent inverters (nots) in electronic diagrams. Note the little circle on the two on the left. The absence of the little circle on the one on the right can leave some ambiguity since the same symbol can be used to represent a non-inverting buffer (gate expander).

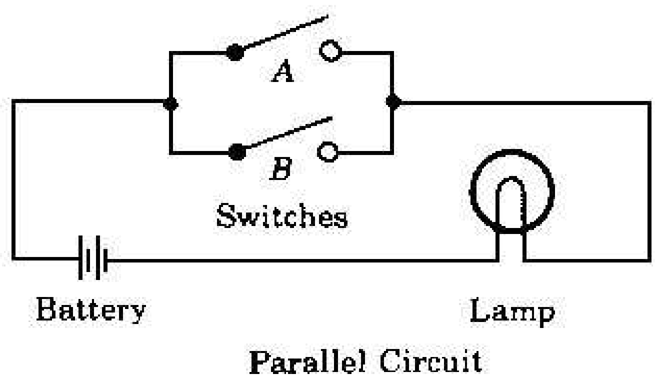


Given below are the corresponding symbols for **and**s, and **or**s. An **and**-gate is equivalent to a series circuit as illustrated in the diagram below right, whereas an **or**-gate is equivalent to a parallel circuit also illustrated below right.









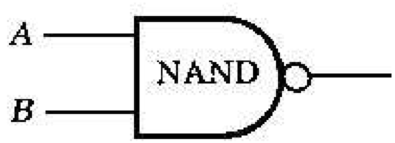
7.5 Nands, Nors, *etc.*

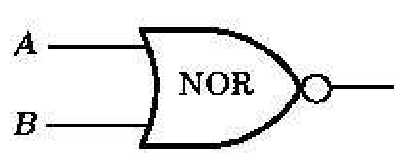
(Earlier we noted p nand q as $\overline{p \wedge q}$ and p nor q as $\overline{p \vee q}$. In the table below we have used the symbol $\overline{\wedge}$ for nand, and the symbol $\overline{\vee}$ for nor.) Compare the **nand**'s and **nor**'s in the table above with those below. Since **nand**'s and **nor**'s can serve as inverters (a not), (by tying both inputs to the source), any logic can be generated using one of them exclusively.

$\overline{\wedge}$	0	1
0	1	1
1	1	0

$\overline{\vee}$	0	1
0	1	0
1	0	0

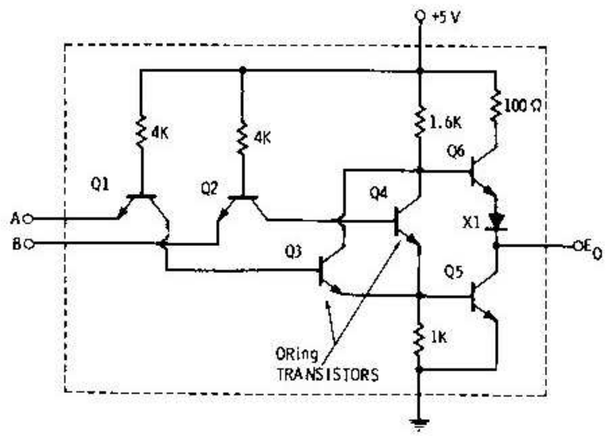
neor	0	1
0	1	0
1	0	1

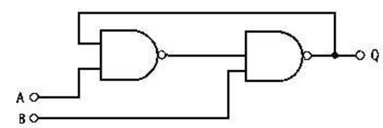




7.6 Logic Equations

Electronic logic was implemented as vacuum tubes ("valves") in the early computers (1950's, generation 1), and with diodes/transistors (DTL, 1960's, generations 2 and 3, with generation 3 being packaged in integrated circuits (ICs)). The 1970's were dominated by TTL (transistor-transistor) logic. In DTL the **nor**-gate was basic whereas in TTL the **nand**-gate was basic. A typical basic **nor** gate and a **nand**-gate based Set-Reset flip-flop are shown below. Many different kinds of flip-flops exist: clocked, D-type, J-K type, J-K master-slave, edge triggered, *etc.*. (Add link here to good electronic site.)

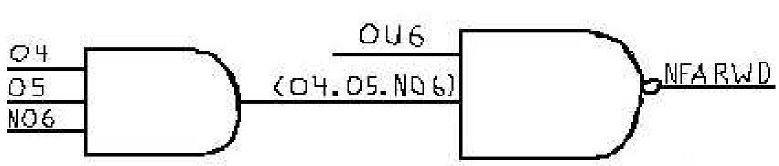




Basic TTL nor gate **Nand-gate Set-Reset flip-flop**

Today, complex microprocessors utilizing millions even billions of logic gates are routinely etched onto silicon chips. However, these basic logic gates composed of several transistors (invented in 1947) are still an important part of the fundamental design. These logic gates are built up into more complex structures such as flip-flops, memory elements, [shift] registers, counters, decoders, multiplexors, adders, *etc.* Often many such microprocessors are etched at the same time on one big silicon wafer. The Pentium 4/D and Core 2 by INTEL now running at speeds of about 4 GHz! demonstrate amazing technological progress.

Some computers of the 1960's and 1970's (SDS/Xerox Sigma) were documented using logic equations. A typical logic equation might read as follows: `NFARWD=I . OU6.(O4.O5.NO6)` (This can be interpret to say that the negation of the signal representing the family of read/write direct instructions (hexadecimal operation codes .6C or .6D) is generated by the upper nibble being a 6 and the lower nibble being .C (lowest order bit being ignored). Here, hex is indicated by the leading period.) A logic diagram is also shown below right. Note how in diagram form this takes up additional space and uses graphic symbols. The logic equation format is very compact and was easily printed using 1960's technology.



7.7 Truth Homework

Each problem is worth two points. For problems 1 and 2 assume these are syllogisms and the major and minor premises are true.

1. Determine whether each argument is valid or invalid. If invalid, determine the error in reasoning.

   (a) If I inherit $1000, I will buy you a cookie. I inherit $1000. Therefore, I will buy you a cookie.

   (b) All cats are animals. This is not an animal. Therefore, this is not a cat.

2. Determine whether each argument is valid or invalid. If invalid, determine the error in reasoning.

   (a) If Alice drinks the water, then she will become sick. Alice does not drink the water. Therefore, she does not become sick.

   (b) If Ron uses Valvoline Motor oil, then his car is in good condition. Ron's car is in good condition. Therefore, Ron uses Valvoline Motor oil.

3. Form a valid conclusion from the following statements.

   (a) If I am tired, then I cannot finish my homework. If I understand the material, then I can finish my homework.

   (b) Everyone who is sane can do logic. No lunatics are fit to serve on a jury. None of your sons can do logic.

4. Form a valid conclusion from the following statement: *No kitten that loves fish is unteachable. No kitten without a tail will play with a gorilla. Kittens with whiskers always love fish. No teachable kitten has green eyes. No kittens have tails unless they have whiskers.*

5. Help Keith find the sugar addict from a truth table of the following statements:
   Keith: Three of you are always right. Who took my oatmeal pie cookies?
   Aurora: It was either Rita or Shirleen.
   Rita: Neither Jenny nor I took it.
   Shirleen: Both of you are wrong.
   Jamie: No, one is wrong; the other is right.
   Jenny: No, Jamie, that's not right.

6. Fill in the truth table:

p	p	$p \barwedge p$	$p \overline{\vee} p$	$\overline{p}$
0	0			
1	1			

7. Construct a truth table for: a) $\overline{p} \wedge q$ b) $\overline{p} \wedge \overline{q}$

p	q	$\overline{p}$	$\overline{q}$	$\overline{p} \wedge q$	$\overline{p} \wedge \overline{q}$
0	0				
0	1				
1	0				
1	1				

8. Construct a truth table for: $[(p \mid q) \wedge \overline{r}] \wedge r$.

p	q	r	$p \mid q$	$\overline{r}$	$(p \mid q) \wedge \overline{r}$	$[(p \mid q) \wedge \overline{r}] \wedge r$
0	0	0				
0	0	1				
0	1	0				
0	1	1				
1	0	0				
1	0	1				
1	1	0				
1	1	1				

Use the following variation on the classic nursery rhyme for the next four questions: *When I was* **coming from** *St. Ives, I meet a man with 7 wives. Each wife had 7 sacks. Each sack had 7 cats. Each cat had 7 kits. Kits, cats, sacks, and wives, how many were going to St. Ives.* (A similar problem dates back to Fibonacci.)

9. Express this quantity in base seven.

10. Calculate the quantity in base ten.

11. Convert the base 7 quantity into base 10.

12. How does this compare with the answer to the traditional wording (**going to**).

13. Construct a "truth table" for the multiplication of positives and negatives.

14. Draw Venn diagrams illustrating DeMorgan's Laws.

15. See Section 2.3 of your geometry textbook for further examples. See especially problems 2.3: 10–13.

16. **Bonus:** Relative to **crore**, find out what numbers the following Hindu terms refer to: **lakh**, neel, padma, shankh.

Numbers Lesson 8

Beyond the Integers: Fractions

Five out of every four Americans has problems with fractions!

Seen on a tee shirt

This lesson moves us beyond the integers by introducing the rational numbers. We explore the axioms which make groups before exploring the parts of a fraction and every type of fraction imaginable. We continue with a review of the addition, subtraction, multiplication, and division of fractions before touching on ratios, proportions, and cross multiplication. First we talk about a mathematician previously introduced.

8.1 Father of Acoustics: Marin Mersenne

Marin Mersenne (1588–1648) was a 17$^{\text{th}}$ century French monk best known for his studies of numbers of the form $2^n - 1$. Mersenne was a well-educated theologian, philosopher, and music theorist. He edited works of Euclid, Archimedes, and other Greek mathematicians. His more important contribution to the advancement of learning was his extensive Latin correspondence with mathematicians and scientists in many countries. Scientific journals had not yet come into being so Mersenne was the center of a network for exchange of information.

Mersenne compiled a list of Mersenne numbers he thought to be prime. His list was only partially correct. It included M_{67} and M_{257} which are composite and omitted M_{61}, M_{89}, M_{107} which are prime. Here we are referring to the number $2^p - 1$ as M_p. It took two centuries to resolve these issues and even yet many fundamental questions about these numbers remain. Questions such as if there is a largest Mersenne prime remain unanswered although it is suspected there are an infinite number of Mersenne primes.

8.2 Group Axioms

It is useful at this time to introduce and discuss the group axioms.

1. Closure: if $\{a, b\} \in G$, then $a \bullet b \in G$ and is unique.
2. Associativity: if $\{a, b, c\} \in G$, then $a \bullet (b \bullet c) = (a \bullet b) \bullet c$.
3. Existence of unit element (identity): $i \in G$, $i \bullet a = a \bullet i = a, \forall a \in G$.
4. Existence of inverses: $\forall a \in G, \exists$ an element denoted $a^{-1} \in G$ such that $a \bullet a^{-1} = a^{-1} \bullet a = i$.

Groups are an important mathematical structure which form the basis of the study of abstract algebra, known to mathematicians as just algebra. The axioms above depend of the concept of a set G with elements a, b, c, *etc.* and one operation ($\bullet$ above) such as addition, multiplication, reflection, *etc.*

Note how the familiar set of natural numbers are closed under both addition and multiplication (axiom 1). Both multiplication and addition are associative (axiom 2), and each has an identity element (axiom 3). The additive identity element is zero (0), whereas the multiplicative identity element is one (1).

Group axiom 4 requires inverses. We have seen our number system "grow" from natural numbers to integers when the operation of subtraction (additive inverses) was introduced. When the operation of multiplication is used and the concept of multiplicative inverses is required, the concept of division is the result and the number system must now include fractions.

An important restriction, the fact that 0 has no multiplicative inverse, will be developed later in Numbers Lesson 9. We thus see that the integers form a group under addition, but not under the operation of multiplication!

8.3 Parts of Fractions

We introduced division in Numbers Lesson 2, but only in the context of integers and remainders.

If you have ever shared an apple with someone, the concept of half should be well developed. Former president George Bush (number 41) was nicknamed "Have half" early in life for this reason. In such a situation, we are dividing one integer by another, often larger, integer.

A **rational number** is a number which can be expressed as the ratio of two integers.

The set of rational numbers is denoted by $\mathbb{Q}$, as in quotient.

A **vinculum** is an overhead line as is used for fractions, radicals, and for repeating decimal fractions. The plural is **vincula**.

The **numerator** is the portion of a fraction above the vinculum.

The **denominator** is the part of a fraction below the vinculum.

Percentage is the numerator of a fraction with a denominator of 100.

Millage or **permille** is the numerator of a fraction with a denominator of 1000.

Percentages are written with a percent sign (%) and permille are written with a permille sign (‰ or ppk). Similar higher order fractions are parts per million (ppm), parts per billion (ppb), and parts per trillion (ppt). Note: there is some ambiguity associated with ppt—it may occasionally represent ppk. These are especially useful for specifying trace amounts or small relative uncertainties.

Example: Lead is a heavy metal which can accumulate in the body. The EPA (Environmental Protection Agency) has set a limit of 15 μg/liter in water which corresponds to 15 ppb since a liter of water has a mass of 1 Kg=1000 g.

Example: Calkins reported in *Physical Review A* 73, 032504 in March 2006 the value 335 116 048 748.1(2.4) kHz for the D_1 centroid for cesium. His uncertainty was thus $2.4/335\,116\,048\,748.1 = 7.2 \times 10^{-12}$ or about 7 ppt. When combined with other measurements it gave a QED-free value for the fine-structure constant $\alpha^{-1} = 137.036\,0000(11)$ or about 8 ppb.

8.4 Types of Fractions

A **unit fraction** is a fraction with a numerator of 1.

Historically, unit fractions were the first to be developed. Ancient Egyptians would add long series of unit fractions to generate other values. It was a historic event when 2/3's came into usage! An application of unit (Egyptian) fractions will be examined in the homework. Today, fractions come in many forms: mixed numbers, improper fraction, **decimal fractions,** *etc.*

An **improper fraction** has a numerator larger (in magnitude) than the denominator, a **proper fraction** does not.

An interpretation of improper fractions is that the denominator says how each whole piece is divided, and the numerator says how many total pieces we have. Improper fractions are quite acceptable in high school and beyond and are, in fact, often the **preferred form of answer.** Too bad elementary/middle school teachers always consider them wrong! However, in their defense, for those less numerically inclined, converting to a mixed number may give a better sense of the number's magnitude. (Converting to a decimal approximation doesn't necessarily do that so clearly!)

A **mixed number** has an integer part and proper fraction part.

A mixed number is generated by dividing the denominator into the numerator to determine how many whole parts there are. The remainder is the numerator of the fractional part.

A **complex fraction** has fractions in the numerator or the denominator.

Partial fractions describes a technique for splitting a fraction into pieces.

This technique will be more formally introduced in Algebra II and is often used in Calculus to simplify a complex expression for ease in integration.

$$\frac{5}{63} = \frac{-49+54}{63} = \frac{-7}{9} + \frac{6}{7} \qquad \text{and} \qquad \frac{5x-1}{x^2-x-2} = \frac{2}{x+1} + \frac{3}{x-2}.$$

Here is an example of a **continued fraction**: $2 + \cfrac{1}{2 + \cfrac{1}{2+\cfrac{1}{2+\dots}}}$

Continued fractions can arise due to recursive definitions. Consider the example above as the solution to the equation: $x = 2 + 1/x$ or $x^2 - 2x - 1 = 0$ or $x = 1 + \sqrt{2}$. Early methods of expressing and extracting square roots depended on this method so it was well developed. It can also be useful for finding integer solutions.

8.5 Operations with Fractions

8.5.1 Simplifying (or Reducing) Fractions

Some examples on how NOT to simplify fractions are as follows:

$$\frac{1\cancel{9}}{\cancel{9}5} = \frac{1}{5} \qquad \text{or} \qquad \frac{1\cancel{3}}{1\cancel{3}0} = \frac{1}{10}.$$

Yet this, perhaps in a slightly more complicated situation, is a very common mistake. Our Algebra II book calls it "freshman cancellation!" Consider what disaster happens when this was done to the examples below. If in doubt, try letting $x = 2$ and compare the before and after results. You can only cancel out factors, where a factor multiplies everything, not terms, where terms are parts of expressions connected by addition and subtraction.

$$\frac{2x}{x+1} \quad \text{or} \quad \frac{x+3}{x-5}.$$

8.5.2 Addition and Subtraction

When adding and subtracting fractions, the first step is to get a **common denominator**. After that the numerators are combined. To get a common denominator, determine the Least Common Multiple. Then multiply each respective fraction's numerator and denominator by a special form of 1 (our multiplicative identity) to get the LCM.

$$\frac{2}{3} + \frac{4}{5}.$$

The LCM $= 15$, so multiply each fraction by 1 so the denominator becomes 15.

$$\frac{2}{3} \cdot \frac{5}{5} + \frac{4}{5} \cdot \frac{3}{3}.$$

Then you add or subtract the numerators, depending on the operation.

$$\frac{10+12}{15} = \boxed{\frac{22}{15}} = 1\tfrac{7}{15}.$$

8.5.3 Multiplying Fractions

To multiply fractions, the rule is to multiply the numerators together and the denominators together. Each product is put in its corresponding location.

$$\frac{10}{11} \cdot \frac{22}{5} = \frac{220}{55} = 4.$$

Of course, after you are done multiplying (or adding, *etc.*), you should always simplify!!! Another way to do it is to **reduce** as you go:

$$\frac{\cancel{10}^{\,2}}{\cancel{11}^{\,1}} \bullet \frac{\cancel{22}^{\,2}}{\cancel{5}^{\,1}} = \frac{4}{1} = 4.$$

8.5.4 Dividing Fractions

In order to divide fractions, reciprocals are useful.

The **reciprocal** of a number is it's multiplicative inverse.

For fractions, this can be obtained by exchanging the numerator with the denominator. The x^{-1} key on the calculator does this as well. Whole numbers are nonnegative fractions with a denominator of 1. (Thus unit fractions are the reciprocals of whole numbers.) Division is equivalent to multiplying by the reciprocal. On many very early computers, this was the only form of division implemented!

Example: $\dfrac{2}{3}$ divided by $\dfrac{1}{6}$. $\qquad \dfrac{2}{3} \div \dfrac{1}{6} = \dfrac{2}{3} \cdot \dfrac{6}{1} = 4$

The reason can be seen by simplifying the complex fraction.

$$\frac{\frac{2}{3}}{\frac{1}{6}} = \frac{\frac{2\cdot 6}{3\cdot 1}}{\frac{1\cdot 6}{6\cdot 1}} = \frac{\frac{2\cdot 6}{3\cdot 1}}{1} = \frac{2\cdot 6}{3\cdot 1} = 4$$

8.6 Ratios and Proportions

Ratios are two numbers with the same units compared. Sometimes they are written like 2:1 or 6:3 where the colon symbolizes that the 2 is compared with 1. Most frequently, ratios are written as division: $2/1$ or $6/3$. When there are more than two numbers involve it is called an **extended ratio.** Here are some examples encountered using ratios:

- An ocean has more water than a lake.
- Enlarging a picture.
- Peter and Paul drove equally fast, but Mary drove twice as far.
- The Tigers are better hitters than the Cubs.
- Tasha is for the metric system because she will be taller in centimeters than in inches.
- The triangle has side length ratios of $3:4:5$.

Proportions are two or more ratios set equal: $\dfrac{2}{6} = \dfrac{1}{3} = \dfrac{12}{36}$. When there are more than two ratios, it is usually called an **extended proportion**. If a proportion has a missing term, we can simply cross-multiply and solve for the missing term.

Example: $\dfrac{x}{16} = \dfrac{1}{4}$ becomes $4x = 16$ which gives $x = 4$.

8.7 Cross-multiplication

Cross-multiplication is actually a short-cut for multiplying each ratio by a special form of 1 involving the other denominator. In other words, you multiply the numerator of one fraction by the denominator of the other and *vice versa* (Latin for **order opposite**; then set these products equal to each other. (See example just above.)

8.8 Fraction Homework

Each problem is worth two points. SHOW WORK, especially on problems 2–6.

1. Use Pascal's Triangle and the Binomial Theorem to expand $(2x+3)^6$ by examining $(2x+3)^2$, $(2x+3)^3$, ...

2. Simplify completely using a common denominator: $\dfrac{1}{7}+\dfrac{1}{11}$.

3. Simplify completely using a common denominator: $\dfrac{1}{7}+\dfrac{1}{13}$.

4. Simplify completely using a common denominator: $\dfrac{1}{11}+\dfrac{1}{13}$.

5. Simplify completely using a common denominator: $\dfrac{1}{7}+\dfrac{1}{11}+\dfrac{1}{13}$.

6. Simplify completely using a common denominator: $\dfrac{9}{143}+\dfrac{18}{77}+\dfrac{8}{91}$.

7. Find 25% of 16.

8. Find 250% of 16.

9. The owner of a house with a state equilized value of $50,000 (the value used for tax computation purposes and which should not exceed half the market value) must calculate how much a proposed 2 mill road improvement tax will cost him. Help him!

10. Express the number 2.7 as: a) an improper fraction; b) a mixed number.

11. Divide 50 **by** $\dfrac{1}{2}$ then add 3.

12. Convert $\dfrac{22}{7}$ **exactly** into a decimal fraction.

13. Simplify completely: $\dfrac{\frac{2}{3}+\frac{1}{2}}{\frac{5}{12}-\frac{1}{4}}$.

14. Simplify completely (factor and cancel common terms): $\dfrac{6}{35}\times\dfrac{15}{22}\times\dfrac{77}{9}$.

15. Simplify completely: $\dfrac{35}{17}\div\dfrac{15}{34}\times\dfrac{6}{7}$.

    For problems 16–18:
    Egyptian fraction is another name for unit fraction. In ancient Egypt, these were the **only** fractions allowed. Other fractions between zero and one were always expressed as a sum of **distinct** Egyptian fractions. The **greedy algorithm** was commonly used to render fractions, such as $\dfrac{3}{5}$, into unit fractions. The algorithm begins by finding two consecutive unit fractions that the given fraction is between ($\frac{1}{2}<\frac{3}{5}<\frac{1}{1}$). Using the smallest fraction, subtract it from the given fraction. This new number plus the smaller fraction is the result. The greedy Egyption number for $\frac{3}{5}$ is $\frac{1}{2}+\frac{1}{10}$ ($\frac{3}{5}-\frac{1}{2}=\frac{6}{10}-\frac{5}{10}=\frac{1}{10}$). Of course, there is no guarantee the result is a unit fraction, so more than 2 fractions may well be required. (See MMPC 1996, part II, problem 1.)

16. Explicitly show how $\dfrac{1}{2}+\dfrac{1}{10}=\dfrac{1}{3}+\dfrac{1}{4}+\dfrac{1}{60}$.

17. Find the greedy representation for $\dfrac{2}{13}$.

18. Find the greedy representation for $\dfrac{9}{10}$.

19. Using your corrected list of the first 15 Fibonacci Numbers from homework 2 problem 3, find the approximate decimal ratio of consecutive pairs. Bonus: what is the **exact** limiting value this approaches?

20. Write the word name for the number which corresponds to $2^{2^5}-1$. Express this number in binary, hexadecimal, and base 10.

21. Read section 11.2 of your geometry textbooks for further examples for Lesson 7. See especially problems 12–17.

Numbers Lesson 9

More on Fractions

No one shall expel us from the Paradise
that Cantor has created. David Hilbert

This lesson presents order of operation for arithmetic, number lines, rules for solving inequalities, and long division. The lesson continues with a discussion of decimal fraction, concentrating further on what makes a fraction repeat or terminate. A section on finding exact rational expressions for repeating decimals is followed with a discussion on division by zero. We conclude the lesson with a proof that the rationals are countably infinite. This proof dates back to Cantor who is featured in a biography.

9.1 Father of Set Theory: Georg Cantor

Georg Ferdinand Ludwig Philipp Cantor (1845–1918) was a German mathematician best known for creating set theory. We will introduce those axioms in Lesson 14. Cantor developed a one-to-one correspondence between various sets but not others. In this way Cantor proved the real numbers uncountable or nondenumerable via a diagonalization argument we will also present in Lesson 14.

Cantor's work raised many philosophic questions and met with serious objections by his fellow mathematicians. Cantor suffered from depression after about age 40, depression likely bipolar in nature, but at the time blamed on the ridicule from his colleagues. Inconsistent proofs due to unclarified assumptions has also been cited as a contributing factor. The philosophic differences especially with Kronecker (See quote at the beginning of Lesson 2) lead to a paradigm shift in mathematics toward using set theory as foundational. The harsh criticism of his work gave way to international accolades by age 60. Long periods of depression limited Cantor's work during the later years of his life with World War I forcing poverty and malnutrition before he died in a sanitarium (mental institution).

Cantor established an unending sequence of larger infinities. Power sets play a key role in this development. He believed his work on transfinite numbers to have been communicated to him by God. Cantor established a one-to-one correspondence between the points on the unit line segment and all the points in an n-dimensional space about which he said: "I see it but I don't believe it!" Cantor is also known for the continuum hypothesis, also discussed in Lesson 14, that no set has more members than the natural numbers and less members than the real numbers.

9.2 Order of Operations

We have already assumed that multiplication occurs before addition and exponentiation before that in Numbers Lesson 5 on bases: $314 = 3 \times 10^2 + 1 \times 10^1 + 4 \times 10^0$. We will summarize these rules here as follows.

1. Operations within symbols of inclusion are done first.
2. Exponentiation is done next **right to left** if stacked.
3. Multiplication and Division are then performed in order **left to right**.
4. Addition and Subtraction are next performed in order **left to right**.

The most common symbols of inclusion are called parentheses ( ), but brackets [ ], braces { }, vincula (plural of vinculum), and others (absolute value, radicals) are also encountered. Some discussion regarding order of exponents is in order. Although mathematicians for centuries have clearly intended $2^{2^3} = 2^8 = 256$ and not $4^3 = 64$, programming languages such as FORTRAN and C and graphing calculators have not been as consistent. The same calculator may be schizophrenic and do it both ways, depending on the circumstances. (Compare $4 \wedge 2^{-1}$ using the x^{-1} key on the TI-84 with $4 \wedge 2 \wedge -1$!)

Be sure to use parentheses whenever encountering stacked exponents.

The rules above are often remembered via the mnemonic (from the Greek meaning a **memory aid**): PEMDAS– **P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally or **P**lease **E**at **M**iss **D**aisy's **A**pple**S**auce. **P**ink **E**lephants **M**ay **D**ance **A**nd **S**way.

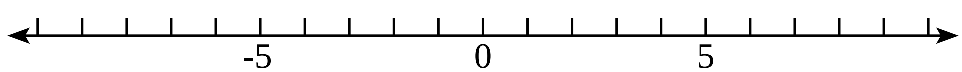
Rule number 3 above deserves a little more ink since really only purists, computer scientists, algebraic calculators, and perhaps high school teachers seem to rigorously adhere to it. Consider expressions such as $3/2\pi$ or $3/2\ \pi$ where **implicit** multiplication might occur. Some textbooks, especially those beyond the high school level, and most high-level math/physics journals assume the 2 is first multiplied by the π in the first example, but not in the second. It is for this reason that I highly recommend against the use of a solidus (/) and for the use of a vinculum (—) especially when handwriting fractions. "No authority decrees this, ...[but] this one rule [multiplication indicated by juxtaposition is carried out before division] is not universal agreement at the present time, but probably is growing in acceptance."[1] When a student answer is an order of magnitude too large I quickly check to see if a π in the denominator wandered "upstairs" due to the lack of parentheses. One can add to this the lack of agreement beyond the high school level in evaluating -1^n. Are we exponentiating negative one, or negating one raised to the n. If n is even, these will differ! Again, purists and calculators following the proscribed order of operations will exponentiate before negating, whereas the other may be intended in some circumstances. This problem originates because the negative symbol ($-$) serves three functions (subtraction, negation, and additive inverse).

9.3 Number Line

A common convention for organizing sets of numbers is to use a **number line**. Some number line conventions will be noted as follows:

1. A number line has larger numbers to the right and smaller numbers to the left. At its center is zero.
2. The integers are usually marked off with tick marks and labelled.
3. Since numbers go on forever, but paper doesn't, arrows are put on each end.

Number lines can be used to show the solution set to certain problems, especially those with infinite solution sets. A sample number line is diagrammed below.

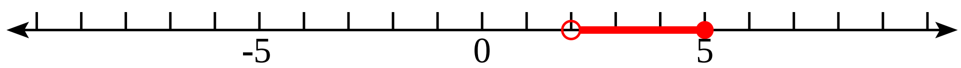


9.4 Inequalities

Mathematics deals not only with equality ($=$) but also with five inequalities $<$, $\leq$, $\neq$, $\geq$, and $>$ known respectively as less than, less than or equal to, not equal, greater than or equal to, and greater than. The big end or opening points toward the bigger quantity. (The alligator is eating the big one, some of my students tell me.) Two of these ($<, >$) are known as the **strict inequalities**, because they do not include the end points. All inequalities but $\neq$ are called **order inequalities**. Number lines are useful to convey such ideas as $x > 2$. To do this, another number line convention should be noted.

[1]Dr. Math: `http://www.mathforum.org/library/drmath/vie/57021.html`

4. If a point is to be excluded at the end of a group of numbers on the number line, an **open** circle is used. Thus, a **closed** circle indicates inclusion of the endpoint. Alternatively, a parenthesis is used to indicate exclusion and a bracket to indicate inclusion. This convention is rooted in the practice of specifying intervals as open, closed, or even half-open, such as $2 < x \leq 5$ as (2,5] shown below.



It should always be clear from context whether an expression such as $(3, 5)$ refers to an ordered pair (See Numbers Lesson 13) or the open interval $3 < x < 5$.

Inequalities are algebraically treated much like equalities (what you do to one side, do also unto the other).

When an inequality is multiplied or divided by a negative number, the direction the inequality points is reversed.

$$\begin{array}{rcll} 1-x & > & 2 & \\ -x & > & 1 & \text{subtract 1 from both sides} \\ x & < & -1 & \text{multiply by } -1 \text{ both sides and reverse the inequality} \end{array}$$

9.5 Long Division

Division is usually the last of the four basic operations $(+, -, \times, \div)$ to be mastered. Division is the inverse operation of multiplication, but has an important exception as discussed below.

The division of one number by another can be represented as a fraction with the dividend as the numerator and the divisor as the denominator. One can simplify the fraction before doing the long division involved.

(Reminder: The divisor is out in front of the "box", the dividend is under it and the quotient is on top of the "box").

$\text{Divisor}\,\overset{\text{Quotient R Remainder}}{\overline{|\text{Dividend}}}$.

An example of a division problem is $441 \div 12$. After reducing, this is the same as $147 \div 4$ or the fraction $\dfrac{147}{4}$. To find the quotient (or to find its mixed number), we divide thusly.

$$
\begin{array}{r|l}
 & \quad 36.75 \text{ (or } 36\ R3 \text{ or } 36\ 3/4) \\
\hline
4 & 147.00 \\
 & \underline{12} \\
 & 27 \\
 & \underline{24} \\
 & \ \ 3\,0 \\
 & \ \ \underline{2\,8} \\
 & \quad 20 \\
 & \quad \underline{20} \\
 & \quad\ 0
\end{array}
$$

9.6 Decimal Fractions

Fractions are often expressed with fairly arbitrary denominators: $\frac{1}{2}, \frac{3}{4}, \frac{2}{3}$. To compare them in magnitude, it is helpful to line them up on a number line: $\frac{1}{2} < \frac{2}{3} < \frac{3}{4}$. To quantify the difference between them, it is helpful to change the denominator to be 10 or a power of ten. Such fractions are called **decimal fractions** or often just **decimals**.

$$
\begin{aligned}
\frac{1}{2} &= \frac{5}{10} = 0.5 \\
\frac{2}{3} &= 0.66666\ldots \\
\frac{3}{4} &= \frac{7.5}{10} = \frac{75}{100} = 0.75
\end{aligned}
$$

So $\frac{2}{3}$ is closer to $\frac{3}{4}$ than to $\frac{1}{2}$. Of course, if we obtained a common denominator of 12, that would have been clear as well: $\frac{6}{12} < \frac{8}{12} < \frac{9}{12}$. The choice of base 10 is very common, although **basimal fractions** related to powers of two are commonly encountered with computers. In fact a marvelous algorithm[2] for calculating π was recently discovered, but involves hexadecimal fractions only.

9.7 Repeating/Terminating Decimal

The number of digits in the repeating unit of a nonterminating but repeating decimal fraction is an area of interesting study. The biggest unit fraction (*i.e.* smallest

[2] `http://www.mathsoft.com/asolve/plouffe/plouffe.html`

denominator) with much interest is $\frac{1}{7} = .142857\ 142857\ \cdots$. As can be seen in the table below, all multiple of $\frac{1}{7}$ have the same digits in the same order, just a different starting point.

$\frac{1}{7}$	$0.\overline{142857}$
$\frac{2}{7}$	$0.\overline{285714}$
$\frac{3}{7}$	$0.\overline{428571}$
$\frac{4}{7}$	$0.\overline{571428}$
$\frac{5}{7}$	$0.\overline{714285}$
$\frac{6}{7}$	$0.\overline{857142}$

In an earlier homework, you already did the equivalent of finding the decimal fraction for 1/7 (7 into 1,000,000; NL1). Note how there can be seven different remainders (0–6) when dividing something by 7. However, if the remainder of 0 is obtained, the fraction terminates (*i.e.* $\frac{7}{7} = 1.0$). This is part of the reason the cycle length is six for the fraction $\frac{1}{7}$. In today's activity you will derive the exact decimal fractions for $\frac{1}{17}$ and $\frac{1}{19}$ which exceed the calculator's accuracy. Of course you could also attempt this by long division like your teacher did since calculators were not common until he was in high school.

Terminating decimals are decimals that have an ending. These numbers do not go on forever or repeat. They are clearly rational numbers since you can express them as the ratio of two integers: the decimal values over the power of ten (what the last digit of the decimal represents). Don't forget to reduce, because this result is not unique. For example, you could multiply the numerator and denominator by 2. **It should be clear that fractions with denominators containing only powers of 2 and 5 (the prime factors of our base 10) terminate, whereas those containing other prime factors do not.**

$$\begin{aligned} 0.115 &= \frac{115}{1000} = \frac{23}{200} \\ 43.336 &= 45\frac{336}{1000} = 45\frac{42}{125} \\ 0.14641 &= \frac{14641}{100000} \end{aligned}$$

9.8 Finding Integer Ratios for Repeating Decimals

Knowing all repeating decimals are rational numbers, or the ratio of two integers, leaves us with the task of finding these integers when presented with an arbitrary example.

Suppose you are asked to find two integers whose ratio is $0.586586\cdots = 0.\overline{586}$. One way is to use the FRAC key on your calculator, but another involves just a little algebra.

Let	$1000x = 586.586586\cdots$
$1000 = 10^3$ was chosen since there are three repeating digits.	
Subtracting off the original	$1x = 0.586586\cdots$
We are left with this	$999x = 586.000\cdots$
or	$x = \dfrac{586}{999}$

For fun, you might try this method on $0.143434343\ldots = 0.1\overline{43} = \frac{142}{990}$!

9.9 Division by Zero

We stated in Numbers Lesson 8 that zero does not have a multiplicative inverse. This is equivalent to the concept that zero multiplied by anything is always zero. If we examine this further, we discover that sometimes things are not quite exactly zero and if multiplied by something big enough, unity will result. Examine the sequence of $0.1\times10 = 1$; $0.01\times100 = 1$; $0.001\times1000 = 1$; $\ldots$ Next examine the same thing but as a division problem: $1\div0.1 = 10$; $1\div0.01 = 100$; $1\div0.001 = 1000$; .... The denomonator approaches zero and the quotient approaches ∞. However, if we approach zero from the other side: $1 \div -0.1 = -10$; $1 \div -0.01 = -100$; $1 \div -0.001 = -1000$; $\ldots$ the result is at the other "end" of our number line. For this reason, it is usual to call division by zero undefined (ill-defined). For some applications, it is useful to join our number line at the two infinities, thus closing our **unbounded** interval! Thus the complete number line (interval between plus and minus infinity) is termed both open and closed.

9.10 The Rationals are Countable

Another important consideration is how many rational numbers are there? The answer may surprise you. Start by listing the natural numbers with one as a denominator. For every successive row, increase the denominator. Then you will have completed a chart containing all the positive rational numbers.

$$\begin{array}{cccccc}
1/1 & 2/1 & 3/1 & 4/1 & 5/1 & \cdots \\
1/2 & 2/2 & 3/2 & 4/2 & 5/2 & \cdots \\
1/3 & 2/3 & 3/3 & 4/3 & 5/3 & \cdots \\
1/4 & 2/4 & 3/4 & 4/4 & 5/4 & \cdots \\
1/5 & 2/5 & 3/5 & 4/5 & 5/5 & \cdots \\
1/6 & 2/6 & 3/6 & 4/6 & 5/6 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Some of them appear more than once ($1/2 = 2/4 = 3/6 = \cdots$). We then count the fractions in this order: $1/1, 2/1, 1/2, 1/3, 2/2, 3/1, \cdots$. Since we have put the natural numbers into a one-to-one correspondance with the positive (unsigned) rational numbers, they are countable or there are "just as many" as natural numbers. This is commonly recognized as the lowest order of infinity, or $\aleph_0$ or aleph null. There are other arrangements possible, such as sorted by "height" (numerator plus denominator) then by numerator, for example. However, fractions cannot be put in a strictly increasing order, because in between each pair is always another! The rational numbers are thus termed **dense**. However, we will see in Lesson 11 there are still gaps between them!

9.11 PEMDAS Homework

Each problem is worth two points, except as noted for problem 12

1. Compare in magnitude the decimal representations for: 22/7, 355/113, and π.

2. Put in order from **least** to **greatest**: $0.1958, 0.195\overline{8}, 0.19\overline{58}, 0.1\overline{958}, 0.\overline{1958}$.

3. Convert $468.468468\cdots$ into the ratio of two integers.

4. Find a rational number between $\frac{2}{7}$ and $\frac{1}{3}$.

5. Simplify: $3 \times 15 + 2 \times 6$.

6. Simplify: $2 \times 6 + 32 \div 4^2 + 5$.

7. Simplify: $4 \times [2 - 3(x + 1)^2] \times (2 - 10 \div 5)$.

8. Solve for x and graph on a number line: $14 - 3x < 13$.

9. Solve for x and graph on a number line: $2x - 4 > -11(x - 2)$.

10. Expand and simplify: $(2x - 3y)(2x + 3y)$.

11. Factor completely: a) $x^2 + 9x + 20$; b) $x^2 + 8x - 20$.

12. Calculate the **exact** decimal representation of the **unit** fractions (See Section 8.4) with denominators 2 through 21. Clearly indicate the length of the part which repeats or whether it terminates (rep.len.=0). Can you find any pattern to the repeat lengths? This problem is worth **17 points**.

Fraction	**Decimal Value**	**Terminates**	**Non-rep. Len.**	**Repeat Len.**
1/2	0.5	yes	1	0
1/3	$0.3333\cdots$	no	0	1
1/4				
1/5				
1/6	$0.1666\cdots$	no	1	1
1/7				
1/8				
1/9				
1/10				
1/11				
1/12				
1/13				
1/14				
1/15				
1/16				
1/17				
1/18				
1/19				
1/20				
1/21				

Numbers Lesson 10

Scientific Notation, Significant Figures, *etc.*

> *An approximate answer to the right question is worth a great deal more than a precise answer to the wrong question.* John Tukey

This lesson is devoted to accuracy, precision, scientific notation, significant figures, and the importance of rounding *vs.* truncating. It ends with sections on common unit of measurement and unit conversions with an emphasis on metric/English equivalents. Our biography for this lesson is on Pascal.

10.1 Miserable Infant Prodigy: Blaise Pascal

Pascal was born, lived, and died in France (1623–1662). He is considered a French philosopher, mathematician, and physicist and one of the greatest minds in western intellectual history. He was the only son of a judge with some scientific background. His early training was restricted to languages and much of his later life was devoted to religious exercises. By age 12 he discovered geometry, read Euclid's *Elements*, and came up with some original proofs. By age 14 he was attending weekly meetings of famous mathematicians, by age 16 he wrote a paper on conic sections, and by age 18 started work on a mechanical adding machine. In correspondence with Fermat he established the theory of probability. This contributed greatly to the development of the fields of actuary, mathematics, social statistics, and physics—not to mention helping his friends with their gambling!

Pascal did research on pressure and invented the syringe. He advocated empirical experimentation and the accumulation of scientific discoveries. Analytic, *a priori* methods were the norm in those days. A run-away horse carriage accident at age 31 further destablized his delicate health and lead him toward religion and away from science and math. The triangle of binomial coefficients, a computer language, a pressure law, and the SI unit of pressure are all named after Pascal.

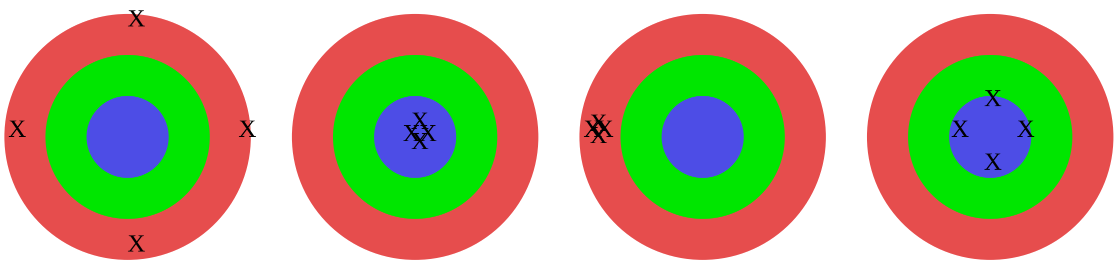


Figure 10.1: Accuracy *versus* Precision Targets. Left: no accuracy, nor precision. Left Middle: accurate and precise. Right Middle: precise, but not accurate. Right: accurate, but not precise.

10.2 Accuracy *vs.* Precision

Accuracy is a measure of rightness.	**Precision** is a measure of exactness.

Versus (*vs.*) is Latin for against or facing. Accuracy and precision, although similar in meaning, have a very subtle difference important to mathematics and science in general and statistics specifically. You can have one without the other, neither, or, best of all, both together. As you can see below, precision has to do with repeatability, how well your results can be reproduced. Here is an example involving e. It is an important number we will study further in Numbers lesson 15. It is also on your graphing calculator is several places.

e=	**Accurate?**	**Precise?**
27	no	no
2.18281828	no	yes
2.72	yes within 1 ppk	no
2.718281828	yes within 1 ppb	yes

Figure 10.1 illustrates what accuracy and precision might mean in the case of a dart board with darts. The table below illustrates the same ideas with words.

Darts	**Accurate?**	**Precise?**
Randomly spread far from the bull's eye	no	no
Clustered inside the bull's eye	yes	yes
Clustered outside the bull's eye	no	yes
Unclustered but inside the bull's eye	yes	no

A common measure of precision is the standard deviation or uncertainty. We will discuss standard deviation more in the upcoming Statistics lectures. **Uncertainty** is the magnitude of error that is estimated to have been made in the determination of results. It is now common to state results in the form: **measurement (uncertainty) units.** Precision can also be thought of in terms of repeatability.

Example: Consider the results from the author's dissertation available at `http://etd.nd.edu`, click on search, enter Calkins as last name, and click now on the search button below. We reported there our 2005 results of the cesium D1 transition centroid frequency as: 335 116 048 748.2(2.4) kHz. Basically, the 2.4 kHz is saying we are about 68% confident that the true value is with ±2.4 kHz of the reported value of 335.116 048 7482 THz (about 894.5 nm or in the infrared).

10.3 Scientific Notation

In science, numbers large and small are commonplace and a shorthand notation call **scientific notation** was developed to simplify their specification and utilization. It is based on place value and base ten. Recall that $10^1 = 10$; $10^2 = 100$; $10^3 = 1000$ and $3 \times 100 = 300 = 3 \times 10^2$ or $10^3 \times 81 = 8.1 \times 10^4$.

A number is in scientific notation if it is in the form: **Mantissa** $\times 10^{\textbf{characteristic}}$, where the mantissa (Latin for makeweight) must be any number 1 through $9.\overline{9}$, and the characteristic is an integer indicating the number of places the decimal moved.

The manissa might sometimes be called a coefficient. The term mantissa is more commonly applied to the decimal fractional portion of a logarithm.

Examples of scientific notation:
92,900,000 miles becomes 9.29×10^7 miles (earth-sun distance).
Planck's Constant: .000000000000000000000000000000000663 Js is 6.63×10^{-34} Js
3141592653 is approximately 3.1416×10^9.
6,600,000,000,000,000,000,000 tons is 6.6×10^{21} (6.6 sextillion) tons or the "mass" of the earth.

Note the use of the **EE key** on calculators and an **E** on computer printouts in reference to scientific notation. 3.14E9 is the same as 3.14×10^9. **D** may also be seen indicating use of **double precision** (typically 64 instead of 32 bits of precision). An easy way to remember when changing number into scientific notation is: if the mantissa is a smaller number in magnitude than your decimal value, then the characteristic will be a positive number. If the mantissa is a larger number than your decimal value, then the characteristic will be negative. Keep this hint in mind as you change from scientific to decimal notation.

Example: $5.43 \times 10^{-3} = 0.00543$, since the characteristic is negative, you know the decimal number is smaller than 5.43, so you move the decimal left. Another example: $-0.000002 = -2 \times 10^{-6}$.

10.3.1 Operations with Scientific Notation

When adding numbers in scientific notation, the characteristics must be the same.

$2.3 \times 10^5 + 4.55 \times 10^3$	The easiest way is to decrease the larger characteristic by rewriting the mantissa!
$230 \times 10^3 + 4.55 \times 10^3 = 234.55 \times 10^3$	After rewriting and adding, rewrite in scientific notation.
$2.3455 \times 10^5 \approx 2.3 \times 10^5$	Results rounded according to rules given below.

Notice what happens when you add the following together: 8.23×10^{17}, 4.67×10^{12}, and -1.05×10^{-12}!

The same method is used when subtracting numbers in scientific notation! Here, however, if they are close in value **loss of significance** may result—the answer may be nonsense! When multiplying numbers in scientific notation, add the characteristics and multiply the mantissas. Division is similar, divide the mantissas and subtract the denominator's characteristic from the numerator's characteristic. **Always convert the answers back into proper scientific notation form.**

Example: $8.1 \times 10^{-3} \times 2 \times 10^5 = 16.2 \times 10^2 = 1.62 \times 10^3$.

Example: $1.08 \times 10^{17} \div 1.2 \times 10^{10} = 0.9 \times 10^7 = 9 \times 10^6$.

A variation on scientific notation is **engineering notation.** In engineering notation the exponent is a multiple of three, reflecting the fact that the standard multiplier in the metric system is $10^3 = 1000$. It is thus more common to speak of meters, kilometers, millimeters, nanometers, and femtometers than is to speak of decimeters and dekameters. Unfortunately, some units such as centimeters and Angstroms are entrenched which complicates our conversion to SI (see below).

Numbers written in scientific notation are **assumed to be measurements,** thus approximations. Therefore, the rules outlined below must be applied.

10.4 Significant Figures, Rounding and Truncating

The significant figures (digits) in a measurement include all the digits that can be known precisely plus a last digit that is likely an estimate.

The rules for determining which digits in a measurement are significant are:

1. Every nonzero digit in a recorded measurement is significant. 24.7 m, 0.743 m and 714 m all have three significant figures.

2. Zeroes appearing between nonzero digits are significant. The measurements 7003 m, 40.79 m, and 1.503 m all have four significant figures.

3. Zeroes in front of (before) all nonzero digits are merely placeholders; they are not significant. 0.0000099 only has two significant figures.

4. Zeroes at the right end of the number **if a decimal point is present** and also zeroes to the right of the decimal (unless leading) are significant. The measurements 1241.20 m, 210.100 m, 0.00123456 m, 5600.00 m, and $12300\overline{0}000$ m all have six significant digits.

5. Zeroes at the end of a measurement and to the left of an omitted decimal point are ambiguous. They are not significant if they are only place holders: 6,000,000 live in New York—the zeroes are just to represent the magnitude of how many people are in N.Y. But the zeroes can be significant if they are the result of precise measurements. A vinculum over the least significant zero is often used.

Examples: tell how many significant figures each of the following has: 9027.0, 9027, 9270, 9270., 0.9270, $927\overline{0}$, and 0.00927.

Solution: 9027.0 has 5 significant digits, 0.00927 has 3. 9270 also has 3 but there is room for doubt. All the rest have 4.

The significant figures in a number in scientific notation is the number of digits in the mantissa. The number 4×10^5 has only one digit in the mantissa, so it has one significant figure. 9.344×10^5 has 4 significant figures. Thus the number 1200 which is unclear as to how many significant figures it has is more clearly expressed as 1.200×10^3 as having 4 significant figures or as 1.2×10^3 as having 2.

When calculating with significant figures, an answer cannot be more precise than the least precise measurement.

This means for...

- **Addition and subtraction**: the answer can have no more digits to the right of the decimal point than are contained in the measurement with the least number of digits to the right of the decimal point.
  Example: 12.21 m + 324.0 m + 6.25 m = 342.46 m, but the answer must be **rounded** to 342.5 m, or 3.425×10^2 m. **Specification of units is also extremely important.**

- **Multiplication and division**: the answer must contain no more significant figures than the measurement with the least number of significant figures (the position of the decimal point is irrelevant).

It is very important to **round** rather than **truncate** your results: $\pi \approx 3.1416$ not $\pi \approx 3.1415$, You are often instructed to round to so many significant digits or to such and such a level of precision. There are variations, but the standard rule would round anything from \$0.50 up to \$1.49 all to \$1. One variation would round \$0.50 down and \$1.50 up based on the evenness/oddness of destination digit. A common mistake to be avoided is "double rounding," for example, rounding 1.46 first to 1.5 and then to 2. More on that will be discuss in the Introduction to Statistics, lesson 3.

10.5 Various Common Units

The National Institute of Standards and Technology, formerly the National Bureau of Standards, is our nation's official source of standard weights and measures, as well as other standards, such as for programming languages. The metric system (Systéme International or SI) has a long, interesting history and is in use the world over. A notable exception is in common (non-scientific) uses in the United States. SI differentiates between basic and derive units and hence is often called the MKS system for meter (length), kilogram (mass), second (time), the fundamental three of the seven basic units. The other four basic units are: K (temperature), ampere (current), candela (illumination), and mole (amount of substance). Listed below is a hodge-podge of units and the most important conversions.

1. English units of volume:
   3 teaspoons=1 tablespoon (useful for child medicine dosage, not just cooking)
   8 tablespoons per stick of butter–4 sticks per pound (Historically, a pound was cut in quarters.)
   2 cups per pint, 2 pints per quart, 4 quarts per gallon, 16 fl oz per pint (a pint's a pound the world round—works only for water. That is, a fluid ounce of water weights about a ounce.)
   231 cu in per gallon (US liquid—there are also Brit and US dry gallons).
   There are 160 Brit oz per Brit Gal., 0.9607594 Brit fluid oz per US fluid oz.
   There are 1.16 US liquid gallons per US dry gallon. 8 US dry gallons per bushel, 4 pecks per bushel. 42 US gallons per US petro barrel (31.5 US gallons per US liquid barrel). 2 US liquid barrels per hogshead. A cord is 4'x4'x8'—be sure to get that and not a third of that ("rick") when buying wood!
   Concrete is specified in cubic yards (27 cu ft per cu yard—why?).
   There are many more "English" units of volume, with a rich history but most are fortunately falling into disuse. I have never had to use: Grains, Scruple (20 grains), Minim (20 scruples), Drachm/Dram (60 minims; 1/8 or 1/16 oz), Gill (5 Brit oz), Bucket (4 Brit gallons), Firkins (9 Brit gallons), Bag (3 bushels), Seam (8 bushels), or Butt (2-4? barrels or 2 hogsheads). Since fresh water on ships was stored in a butt, and people congregated and gossiped there, the term scuttlebutt now refers to gossip, not just the fountain!

   Note: 33.8 ml/fl oz and 3.785 liters per gallon are useful crossovers.

2. Common "English" units of weight include: caret (200 mg), ounce (12 apothecaries/troy or 16 avoirdupois per pound!), pound, and ton (2000 pounds per short ton, 2240 pounds per long ton, 2204 pounds per metric ton). Mostly fallen into disuse are: pennyweight (20 per troy oz), slug (32.174 avdp. pounds),

hundredweight (20 per ton). **Pounds are, of course, abbreviated as lb!**

28.349523 grams per ounce and 2.20 pounds per kilogram are useful crossovers. Also, a nickel weights exactly 5 grams and a post-1982 penny half that.

3. Common units of time are: the pico-, nano-, micro-, milli-, seconds. There are 60 seconds per minute (angle or time!), 60 minutes per hour (or degree), 24 hours per day, 7 days per week, 14 days make a fortnight, 365.24 days per year more or less. There are sidereal, calendar, and tropical years as well as calendar and lunar months. We also speak of decades, centuries, millenia, age of the earth (4.5 billion years), or universe (about 13.7 billion years=a Hubble time).

   NIST is responsible for defining the second, currently via the cesium fountain clock and cooperates internationally to generate world time known as Coordinated Universal Time (UCT). However, the US Navy is responsible for maintaining and distributing this time and uses several dozen cesium clocks and about one dozen hydrogen masers to do this. They are researching the use of a cesium fountain clock to help stabilize and steer the hydrogen masers. The second is metric. The 21st century/3rd millennium started January 1, 2001. Also, the designations 12 am (technically noon, Chicago style midnight) and 12 pm should not be used.

4. You are responsible to know and understand the metric prefixes of: Giga, Mega, Kilo, milli, micro, nano, and pico. You should be very aware that giga(G), mega(M), and kilo(K) can have slightly different meanings especially when used in a computer related context. There K refers not to 1000, but to $1024 = 2^{10}$. M might refer to 1,000,000; 1,024,000 (3.5" floppies!); or $1048576 = 2^{20}$. G might refer to 1,000,000,000; $1,073,741,824 = 2^{30}$; or possibly some number in between! The terms Kibi(Ki), Mebi(Mi), Gibi(Gi) have been suggested.

5. Common "English" units of length include the inch, foot (12 inches per foot), yard (36 inches per yard), mile (5280 feet per statute mile—a nautical mile is about 6076 feet (Int) or 6080 feet (Brit)). My father still speaks in rods (16.5 feet), which is also a pole or perch. Physicists speak of lightyears (5.8785×10^{12} miles or 9.46×10^{12} km). This is the distance light travels in one year. **Light in vacuum travels exactly 299,792,458 meters per second (about 3×10^8 m/s).** This value is c. When combined with the definition of the second, this **defines** the meter. Hands (4"), mil (.001"), and points (about 1/72") are still commonly used. Falling into disuse are furlongs (8 per mile), leagues (3

naut. miles), fathom (6 feet), chains (80± per mile), and cables (720 feet).

More metric crossovers: Exactly: 2.54 cm/in= 39.37 inches per meter.
Approximately 1.609 km per mile or 0.62 miles per km.

Feet are often abbreviated as single quotes and inches as double quotes. (I am 5'6".) These same quote symbols are used for angle measurement in minutes, seconds, and thirds. (A right angle is $90°0'0''0'''$.)

10.6 Unit Conversions

Converting from one type of unit to another is a common occurance in science. It is just another incidence of multiplying by our multiplicative identity (1)! For example, to convert 0.62 miles into feet we multiply by the identity 5280 feet/1 mile. The units of miles in the numerator and demominator cancel and we are left with 3273.6 feet. (More than 3 significant figures were retained, since 5280 is an exact value.) Two additional and useful conversions are given below as further examples.

Example: 60 miles/hour × 5280 ft/mile × 1 hour/3600 s = 88 ft/s.

Example: 5280 ft/mile × 5280 ft/mile /640 acres/sq mile=43560 sq ft/acre. This is a square about 209 ft on a side or a rectangle exactly $132' \times 330'$. A square mile is a section, 36 sections are a geographic township. Political townships vary in size.

10.7 Significant Figures Homework

Each problem is worth two points, except as noted.

1. Using your TI-84 calculator result for 69! in scientific notation, multiply 69! by 7 and approximate 70! also in scientific notation.

   For problems 2–7 **round** each measurements to the number of significant figures shown in parentheses. Write your answer in **scientific notation.**

2. 314.721 m (4 sig. fig.)

3. 0.001775 m (2 sig. fig.)

4. 64.32×10^{-1} m (1 sig. fig.)

5. 8792 m (2 sig. fig.)

6. 87.073 m (3 sig. fig.)

7. 4.3621×10^{8} m (1 sig. fig.)

   For problems 8–17 do the following operations and give the answer to the **correct number of significant figures.**

8. 74.626 m − 28.34 m

9. 61.2 m + 9.35 m + 8.6 m

10. 9.44 m − 2.11 m

11. 1.36 m + 10.17 m

12. 34.61 m − 17.3 m

13. 2.10 m × 0.70 m

14. 2.4526 m ÷ 8.4.

15. 0.365 m ÷ 0.0200.

16. $(1.8 \times 10^{-3}\,\mathrm{m}) \times (2.9 \times 10^{-2}\,\mathrm{m})$
17. $5.3 \times 10^{-2}\,\mathrm{m} \div 0.255$
18. (**Four points:**) The five students at table #2 obtained the following measurements for the length of 12-inch rulers in centimeters (four groups tried it 4 times). Determine whether each student's measurements were accurate and/or precise.

Meas. #\Student:	**Audrey/Rashmi**	**Becky**	**Cami**	**Kara**
1.	31.51	30.4	30.281	28.1
2.	31.45	30.5	30.781	28.9
3.	31.61	30.3	30.441	28.7
4.	31.35	30.4	30.431	28.6
Accurate:	Yes or No	Yes or No	Yes or No	Yes or No
Precise:	Yes or No	Yes or No	Yes or No	Yes or No

19. (**Three points:**) Identify the exponent for the power of ten multiplier for each of the following metric prefixes. (Hint: they are in order and all the missings ones are multiples of three.)

Prefix	$\times 10^?$
Yotta-	24
Zetta-	21
Exa-	18
Peta-	15
Tera-	
Giga-	
Mega-	
Kilo-	
deci-	-1
centi-	-2
milli-	
micro-	
nano-	
pico-	
femto-	
atto-	-18
zepto-	-21

20. **Bonus:** Find a humorous unit/prefix such as 10^6 phones is one Megaphone!

Numbers Lesson 11

Beyond Rationality

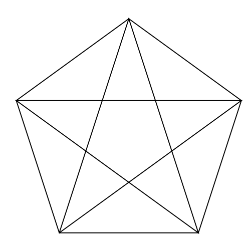
All is number. Pythagoras

In this lesson we will explore numbers which cannot be expressed as the ratio of two integers, *i.e.* irrational numbers. Our biography is on Pythagoras and then we explore a proof often attributed to him that many radicals are irrational. We study the parts of a radical and how to simplify and multiply them. We discuss rationalizing denominators and give the old method of extracting roots by hand. We close with a section on the Golden Ratio.

11.1 The Father of Numbers: Pythagoras

Pythagoras was an ancient Greek (c. 576–c. 500 B.C., both dates have large single digit uncertainties) mathematician, philosopher, and mystic perhaps best known for his theorem and school. We will discuss the Pythagorean Theorem in the next lesson. Many mathematical results are attributed to Pythagoras but some of them were likely developed by his students at his school/brotherhood, a few even after he died. At this time it is very difficult to separate the man from his legend.

Pythagoras coined the word **philosophy** to signify a love of wisdom. Pythagoras and his school believed everything could be described mathematically, hence predicted and measured. Rhythmic cycles were often involved, especially in describing the **cosmos**, another word he likely created. Mathematics and religion thus became comingled. Thought became superior to observation, a notion still present in many religions with an antiscience bias.



Pythagoras was born on an island off Greece settled by Greeks. His secret religious school was communal (at least for those in the inner circle) and lasted several generations after his death, thus influencing Aristotle, Socrates, and Plato. Secrecy was not always well observed. The school was located in southern Italy. Both male and female students were welcome and treated equally at a time when women were often considered property. The pentagram (a regular pentagon with all diagonals producing a 5-pointed star) was their symbol. Any writings Pythagoras produced did not survive, but his teachings may have all been strictly oral.

In astronomy the known planets were said to produce a harmony of the spheres. Musical tones and scales were also studied. One story has his school studying the blacksmith's anvils which harmonized because of their simple proportional sizes. Pythagoras believed in reincarnation and claimed to remember four previous lives. Many of his followers or disciples studied in Egypt where the transmigration of the soul was a common belief. Pythagoras was also the first influential Western vegetarian. Beans were also not to be eaten since they contained or transmitted souls, although it is possible abstaining from beans really meant abstaining from politics. Pythagoras's death may have been a murder and some tales indicated he stopped running when he came to a field of beans.

11.2 Irrational Numbers

It was widely believed that all numbers were rational, expressible as the ratio of two integers, until the Pythagorean school (around 500 B.C.) discovered otherwise. (Legend has it that someone shared this secret ("spilled the beans") and was thrown overboard the ship they were on at the time.) Today, such numbers are called irrational numbers. Since then irrational has become an adjective meaning lacking normal logical clarity! The square root of 2 ($\sqrt{2}$) may have been the first irrational number discovered. It is the solution to the simple problem $x^2 = 2$.

Irrational numbers are real numbers that cannot be expressed as the ratio of two integers.

Common irrational numbers are nonrepeating and nonterminating decimals. These include the roots of any prime and indeed most radicals.

11.3 Simplifying Radicals

The symbol $\sqrt[n]{\ }$ is called a **radical**. The number underneath the **surd** symbol ("check-mark") is the **radicand**. n is the **root index**, indicating what the root is. When no root index appears, 2 meaning square root is assumed.

Irrational numbers were originally considered absurd! Historically radicals were written without a vinculum: $\surd(2)$, for instance.

$\sqrt{2}$ can also be written as $2^{\frac{1}{2}}$. In general, $x^{a/b}$ means the b^{th} root of x^a. Such rational exponents still follow the exponentiation rules given in Numbers Lesson 5.

11.4 Pythagoras's Proof that the $\sqrt{2}$ is irrational

Given below is a proof often attributed to Pythagoras of the existence of irrational numbers using the $\sqrt{2}$ as an example. (Some have suggested that the golden ratio was the first irrational number discovered.)

Statements	**Reasons**
$\sqrt{2} = a/b$	Proof by contradiction: assume true what we are proving false
$2 = a^2/b^2 \qquad 2b^2 = a^2$	Square both sides (expressions remain equal)
a and b have no common factors	assumed without loss of generality: a/b represents reduced fraction
If a is odd, a^2 is odd, but $2b^2$ is clearly even, a contradiction	odd times odd is odd, a cannot be both even and odd simultaneously.
If a is even, let $a = 2c$	even can be factored into 2 and another number even (2) times anything is even
$a^2 = a \cdot a = 4c^2 = 2b^2$	Substitution of equals into product (twice)
$2c^2 = b^2$	Division Property of Equality
So b is even; hence a, b have the common factor 2, a contradiction.	Q.E.D. (*quod erat demonstrandum:* Latin for which was to be proved.)

When simplifying radicals, break the radicand into factors of perfect squares, cubes, *etc.* (9 is the perfect square of 3, 4 is the perfect square of 2, 27 is the cube of 3). Separate the factors into separate radicals. Then express the roots of the radicals with perfect squares, cubes, ...

Examples:

$$\sqrt{27} = \sqrt{9 \cdot 3} = \sqrt{9} \cdot \sqrt{3} = 3\sqrt{3}$$

$$\sqrt{96} = \sqrt{16 \cdot 6} = \sqrt{16}\sqrt{6} = 4\sqrt{6}$$

$$\sqrt[3]{250} = \sqrt[3]{125 \cdot 2} = \sqrt[3]{125}\sqrt[3]{2} = 5\sqrt[3]{2}$$

11.5 Multiplying Radicals

When multiplying radicals, multiply the radicands of like root indexes and then simplify the product. Usually, the easiest way is to simplify as you go along so that you don't end up with large products to factor.

Examples:

$$\sqrt{6}\sqrt{3} = \sqrt{6 \cdot 3} = \sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}$$

$$(\sqrt{7})^2 = \sqrt{7}\sqrt{7} = 7$$

$$(2\sqrt{5})^2 = 2\sqrt{5} \cdot 2\sqrt{5} = 4 \cdot 5 = 20$$

Compare the next two examples and notice how they differ. Both methods are correct. Choose the one which saves you the most time.

$$\sqrt{50}\sqrt{15} = \sqrt{750} = \sqrt{25 \cdot 30} = 5\sqrt{30}$$

$$\sqrt{50}\sqrt{15} = 5\sqrt{2} \cdot \sqrt{15} = 5\sqrt{30}$$

Note when the radicals have different root indexes:

$$\sqrt[3]{16}\sqrt{2} = \sqrt[3]{8 \cdot 2}\sqrt{2} = 2\sqrt[3]{2} \cdot \sqrt{2}$$

11.6 Rationalizing Denominators

Common practice is to simplify expressions to get rid of radicals in the denominator of fractions. Historically, this was all but necessary before calculators. (Imagine dividing $\sqrt{2}$ by the $\sqrt{3}$ by long division!) In order to rationalize the demoninator, the common practice of multiplying by one is used. One comes in many forms: anything divided by itself is one. So multiply the fraction by the square root that is in the denominator over itself.

Examples:

$$\sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{3}\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{6}}{2}$$

$$\sqrt{\frac{16}{12}} = \frac{4}{2\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\sqrt{\frac{1}{8}} = \frac{1}{2\sqrt{2}} = \frac{1\sqrt{2}}{2\sqrt{2}\sqrt{2}} = \frac{\sqrt{2}}{4}$$

11.7 Extracting Roots

The $\sqrt{2}$ can be approximated on your calculator. Before calculators were developed, the following method was widely taught and used. It is based on Newton's Method which will be taught in calculus. Since the decimal representation of $\sqrt{2}$ goes on forever without terminating or repeating, calculators can only give you a fairly precise decimal approximation.

> Whenever you use the decimal approximation of a radical, you should note that it is an approximation and not exact by the use of the symbol $\approx$.

1. Separate the number into groups of two digits going each way from the decimal point.

2. Estimate the largest square which will go into the first group.

3. This number goes both in the normal divisor's location for long division and above the first group as in long division.

4. Double this digit and bring it down for the next step (see example below).

5. Also bring down the next group of digits as in long division.

6. Estimate how many times the two digit number formed using this doubled digit and the number of times...will go into the number.

7. Repeat steps 4–6 above, but now the number down will be 2, 3, 4 digits, *etc.* Continue until the desired accuracy is achieved.

Example: Extracting root 2.

Step 1: $$\begin{array}{ll} & ?.\ \ ?\ \ ?\ \ ?\ \ ?\ \ ?\ \ ? \\ ?\ \ / & \overline{2.\ 00\ 00\ 00\ 00\ 00\ 00} \end{array}$$

Find an integer that squared goes into 2:

Step 2: $$\begin{array}{ll} & 1 \\ 1\ \ / & \overline{2.\ 00\ 00\ 00\ 00\ 00\ 00} \end{array}$$

Double the quotient and bring down to be the divisor. Another digit will follow.

$$\begin{array}{ll} & 1.\ \ ? \\ 1\ \ / & \overline{2.\ 00\ 00\ 00\ 00\ 00\ 00} \\ & \underline{1\qquad} \\ 2?\ \ / & 1\ \ 00 \end{array}$$

Find the number,?, so that 2? will go into 100 ? times. (We find that it is 4: $24 \cdot 4 < 100 < 25 \cdot 5$)

$$\begin{array}{rcl}
 & & 1.\ \ 4 \\ \cline{3-3}
1 & / & 2.\ 00\ 00\ 00\ 00\ 00\ 00 \\
 & & \underline{1\quad\ } \\
24 & / & 1\ 00 \\
 & & \ \ \ \underline{96} \\
 & & \ \ \ \ \ 4
\end{array}$$

We continue to repeat the steps: double the quotient and find the last digit until we get the precision we need.

$$\begin{array}{rcl}
 & & 1.\ \ 4\ \ 1 \\ \cline{3-3}
1 & / & 2.\ 00\ 00\ 00\ 00\ 00\ 00 \\
 & & \underline{1\quad\ } \\
24 & / & 1\ 00 \\
 & & \ \ \ \underline{96} \\
281 & / & \ \ \ \ \ 4\ 00 \\
 & & \ \ \ \ \ \underline{2\ 81} \\
 & & \ \ \ \ \ 1\ 19
\end{array}$$

How long would it take you to verify for accuracy the following level of precision?[1]

$\sqrt{2} = 1.41421\ 35623\ 73095\ 04880\ 16887\ 24209\ 69807\ 85696\ 71875\ 37694\ 80731$ $76679\ 73799\ 07324\ 78462\ 10703\ 88503\ 87534\ 32764\ 15727\ 35013\ 84623 \cdots$.

11.8 Golden Ratio

Another curious irrational number is $\Phi = \frac{1+\sqrt{5}}{2} \approx 1.618\cdots$ and his partner $\Phi' = \frac{\sqrt{5}-1}{2} \approx 0.618\cdots$. These are known as the **Golden Ratio** and symbolized by Φ, the Greek letter capital phi. Notice how things like $3'' \times 5''$ cards often assume these proportions. Notice also how ratios of consecutive Fibonnaci numbers approach the Golden Ratio as seen in Numbers Homework 8.8. The Golden Ratio is also one of the roots of the quadratic equation $x^2 - x - 1 = 0$. If you change the 2's in the continued fraction given in Numbers Lesson 8 to 1's, you will have yet another representation!

$\Phi = 1.61803\ 39887\ 49894\ 84820\ 45868\ 34365\ 63811\ 77203\ 09180\ \cdots$

[1]WARNING: some students have naively programmed this on their calculator and not gotten this result due to round off error.

Name ____________________ Score ________

11.9 Radical Homework

Each problem is worth one point.
Examples: Simplifying Square Roots

$\sqrt{75} = \sqrt{25 \cdot 3} = \sqrt{25} \cdot \sqrt{3} = 5\sqrt{3}$
$\sqrt{76} = \sqrt{4 \cdot 19} = \sqrt{4}\sqrt{19} = 2\sqrt{19}$
$\sqrt{144} = \sqrt{9 \cdot 16} = \sqrt{9}\sqrt{16} = 3 \cdot 4 = 12$ $\quad \sqrt{54} = \sqrt{9 \cdot 6} = \sqrt{9}\sqrt{6} = 3\sqrt{6}$ not $3\sqrt{2}\sqrt{3}$

Examples: Multiplying Square Roots

$(\sqrt{3})(\sqrt{2}) = (\sqrt{6})$
$(\sqrt{3})^2 = (\sqrt{3})(\sqrt{3}) = 3$
$(2\sqrt{3})^2 = (2\sqrt{3})(2\sqrt{3}) = 4 \cdot 3 = 12$

Examples: Rationalizing the Denominator

$\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$
$\sqrt{\frac{3}{8}} = \frac{\sqrt{3}}{\sqrt{8}} = \frac{\sqrt{3}}{\sqrt{8}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{\sqrt{16}} = \frac{\sqrt{6}}{4}$

Express each square root EXACTLY in simplest form (one point each).

1. $\sqrt{12}$ 2. $\sqrt{18}$ 3. $\sqrt{24}$ 4. $\sqrt{32}$ 5. $\sqrt{40}$

6. $\sqrt{48}$ 7. $\sqrt{60}$ 8. $\sqrt{75}$ 9. $\sqrt{73}$ 10. $\sqrt{95}$

11. $\sqrt{90}$ 12. $\sqrt{216}$ 13. $\sqrt{120}$ 14. $\sqrt{235}$ 15. $\sqrt{810}$

16. $\sqrt{324}$ 17. $\sqrt{720}$ 18. $\sqrt{242}$ 19. $\sqrt{784}$ 20. $\sqrt{828}$

Express each product EXACTLY in simplest form.

21. $(3\sqrt{2})^2$ 22. $(4\sqrt{3})^2$ 23. $(2\sqrt{3})(\sqrt{2})$ 24. $(3\sqrt{6})(2\sqrt{3})$ 25. $(7\sqrt{3})^2$

Rationalize the denominator, then simplify EXACTLY.

26. $\sqrt{\frac{1}{3}}$ 27. $\sqrt{\frac{5}{24}}$ 28. $\sqrt{\frac{7}{27}}$ 29. $\sqrt{\frac{35}{50}}$ 30. $\sqrt{\frac{1}{2}}$

Numbers Lesson 12

Theorems: Pythagorean, Fermat's Last, *etc.*

I have discovered a truly marvelous demonstration which this margin is too narrow to contain. Pierre de Fermat

This lesson introduces two important theorems, the Pythagorean Theorem and Fermat's Last Theorem (FLT). We repeated the quote above due to its importance. Considerable space is given to an introduction to trigonometry before the Pythagorean Theorem is applied to the practical application of finding distances. Diophantine analysis is introduced to help discuss FLT. Perfect cuboids, the Fermat-Catalan Conjecture, and Goldbach Conjecture are also covered.

12.1 The Father of Modern Mathematics: Fibonacci

The Italian Fibonacci or Leonardo of Pisa (*c.* 1170–*c.* 1250) was the "most talented mathematician of the Middle Ages." Fibonacci is best known for spreading the use of the Hindu-Arabic place value number system and also a sequence of natural numbers presented earlier. The name Fibonacci may have been assigned posthumously or was the name Leonardo published under. In either case it seems to be a reference to his father and some have suggested it to be self-depreciating in that his father's nickname meant simple. Leonardo's father was a merchant and thus he visited Arab markets in North Africa and as a young boy Leonardo learned the computation methods there. Leonardo's publication caused the eventual displacement of the use of Roman numerals thus ushering in modern arithmetic. The Fibonacci sequence was not new with Fibonacci, but his publication of it in conjunction with the tallying of a rabbit population popularized it.

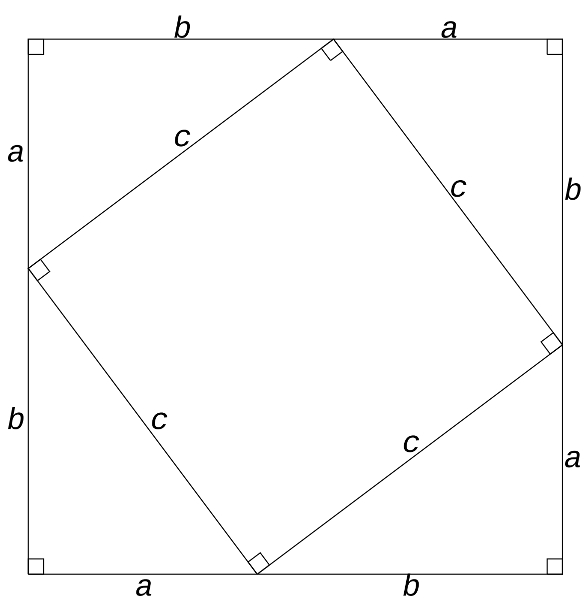
12.2 Pythagorean Theorem, Proof, Triples

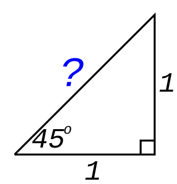
One of the most important discoveries in antiquity was that not only did $3^2+4^2=5^2$, but also, if such a triple could be found, these were the side lengths of a **right triangle**. (A right triangle contains one 90° or right angle.) Several cultures (Chinese, Babylonians, Egyptians, and Greeks) may have independently made this discovery, but due to our historic European slant and records preservation, this has been known as the Pythagorean Theorem. However, the Greeks went further, developing geometry not only for practical purposes, but also in abstraction and for its logical structure. The Pythagorean Theorem is one of the most important facts learned in Geometry.

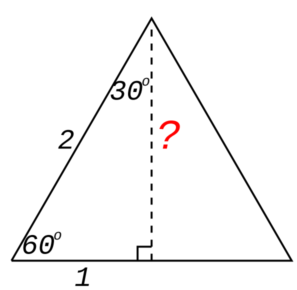
A triangle with sides a, b, and c (longest) is a right triangle if and only if $a^2+b^2=c^2$.

Hence we know how the sides are related if it is a right triangle. We can also prove the triangle to be a right triangle if its sides have this relationship—the converse situation.

There are over three hundred different proofs of the Pythagorean Theorem. One of the common proofs uses a square within a square (see figure below). Each side of the inner square has length c. Each corner of the inner square intersects the sides of the outer square. The four triangles formed by the intersection are all congruent. Therefore each side of the outer square is made up of two segments, a and b.







In order to find the distance c in terms of a and b, we use the fact that the area of the outer square is the same as the sum of the area of the four triangles and the inner square. The rest is algebraic manipulation. $(a+b)^2 = c^2 + 4(\frac{1}{2})ab$. Expanding, we get: $a^2+2ab+b^2 = c^2+2ab$. After subtracting $2ab$ from both sides, we conclude that $c^2 = a^2 + b^2$. Q.E.D.[1]

A **pythagorean triple** is a set of three integers a, b, c such that $a^2 + b^2 = c^2$.

A **primitive pythagorean triple** is a pythagorean triple such that GCF$(a, b) = 1$.

Common pythagorean triple are: 3, 4, 5; 5, 12, 13; 7, 24, 25; 9, 40, 41; and 6, 8, 10. All but this last triple are primitive. The last is called a **multiple**. Note: it follows that if GCF$(a, b) = n$, then n is also a factor of c. Notice how $3^2 = 4 + 5$; $5^2 = 12 + 13$, ... This is a characteristic of a general class of primitive pythagorean triples involving squares and two consecutive integers and was illustrated in homework 3, problem 6. Pythagorean triples such as 8, 15, 17 do not have this characteristic.

12.3 Special Triangles

A **regular polygon** has all sides equal (**equilateral**) and all angles equal (**equiangular**). In a triangle these cannot occur independently. The resulting triangle with sides in the ratio 1:1:1 and angles of $60°, 60°, 60°$ is discussed, in part, below. The three most important right triangles are: the $3, 4, 5$; the isosceles right $(45°, 45°, 90°)$; and the $30°, 60°, 90°$ triangle. The $3, 4, 5$ triangle has angle measures of about $37°$, $53°$, $90°$. Watch especially for these special angles and triangles.

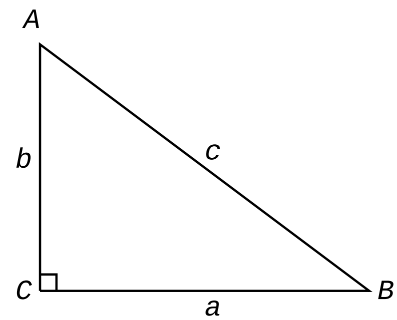
The **isosceles** (2 or more sides equal) **right** (having a $90°$ angle) triangle can be thought of as having **legs** (the shorter sides of a right triangle) of length 1. Thus the **hypotenuse** (the longest side of a right triangle) is $\sqrt{1^2 + 1^2} = \sqrt{2}$. Please label the upper "?" (blue ?) thusly in the figure above. The $30°, 60°, 90°$ triangle can be thought of as a bisected[2] equilateral triangle. Thus one side might be 1, the hypotenuse then is 2 and the other side must satisfy $1^2 + x^2 = 2^2$, or $x^2 = 3$, thus $x = \sqrt{3}$. Please label the lower "?" (red ?) thusly in the figure above. These side length ratios must be memorized and will be seen often in trigonometry which is the study of triangle measure, but primarily involves triangle side length ratios. Note: if $a^2+b^2 < c^2$, the triangle is **obtuse** (contains an angle more than $90°$). If $a^2+b^2 > c^2$, the triangle is **acute** (all three angles are less then $90°$).

[1]We realize this proof depends on the concept of area and the area formula for triangles, items not yet formally covered in this course. Motivation for them could occur back when factors are presented.

[2]We will formally define this term in Geometry, but its meaning should be clear here: to cut into two equal parts.

12.4 Trigonometry Definitions

A quick introduction to a semester of trigonometry can be summarized as follows. Three items taken two at a time can be done six different ways (${}_3P_2 = 3!/(3-2)! = 6/1 = 6$). One trigonometric definition involves ratios (two numbers) of the three sides of a right triangle. For sake of future reference, we will identify the triangle as $\triangle ABC$ with right angle C. This is a very standard convention. Side c is then the hypotenuse and is opposite $\angle C$, *etc.* In relation to angle A, a is its **opposite** side and b is its **adjacent** side (adjacent means **to lie nearby**). See the figure to the right.



$\boxed{\sin A\text{=opposite/hypotenuse} \ \cos A\text{=adjacent/hypotenuse} \ \tan A\text{=opposite/adjacent}}$

sin is the normal abbreviation for **sine** and in English is pronounced the same with a long i sound (saying its name). It comes from the Latin word for **curve** which came from a Sanskrit word meaning **bowstring**. **cos** is the normal abbreviation for **cosine** where the prefix **co-** has the usual meaning of **together** or **partner**. **tan** is the normal abbreviation for **tangent** from Latin meaning **to touch** which has a more general geometric meaning of the intersection of two geometric figures at a point. These relationships are often remembered via the mnenomic **SOH CAH TOA**. One can readily see that tan A=sin A/cos A. The remaining three trigonometric functions: **secant** or **sec** A=1/cos A; **cosecant** or **csc** A=1/sin A; and **cotangent** or **cot** A=1/tan A are less frequently used and usually don't even appear on calculators. Remember, there is only one cofunction in each reciprocal relationship. It is important to note that a rather confusing notation is historically used for the inverse trigonometric functions. $\sin^{-1} x$ refers not to the reciprocal of $\sin A$, but rather to the inverse function. That is $\sin^{-1} x$ is an angle whose sin is equal to x. However, $\sin^2 x$ means $(\sin(x))^2$ and must be entered as such on your calculator. The table below follows directly from these special triangles and trigonometric definitions.

$\tan 90^\circ$ is ill-defined since $\cos 90^\circ = 0$ (or the adjacent side is zero) and division by zero is not allowed. More will be presented on the trigonometric function definitions after Number Lessons 13 introduces the cartesian coordinate system and Number Lesson 15 introduces transcendental numbers.

Angle (deg)	Angle (Radians)	Sine	Cosine	Tangent
$0°$	0	0	1	0
$30°$	$\frac{\pi}{6}$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
$36°52'11.63\ldots''$	$0.64350\ldots$	$3/5$	$4/5$	$3/4$
$45°$	$\frac{\pi}{4}$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$53°7'48.36\ldots''$	$0.92729\ldots$	$4/5$	$3/5$	$4/3$
$60°$	$\frac{\pi}{3}$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$90°$	$\frac{\pi}{2}$	1	0	ill-defined

12.5 Distance

The most important applications of the Pythagorean Theorem is for finding the distance between points in a plane. See Numbers Lesson 13 for the formal development of the cartesian coordinate system. Consider the points $(1,2)$ and $(4,6)$. Since our x and y axes are **orthogonal** (as in at right angles or mutually **perpendicular**), it should be clear that the distance between them is $\sqrt{4-1^2+6-2^2} = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25}$, which is 5. In general, the distance between two points (x_1, y_1) and (x_2, y_2) is:

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Points 1 and 2 may be interchanged with no affect since the squaring operation forces the result positive. That is, distance is always positive, unless termed **directed distance**, in which case it may be negative.

12.6 Diophantine Analysis

Integers were the first numbers to be discovered and studied. As a result, considerable efforts went into finding integer solutions to some problems. Diophantus of Alexandria, a Greek, lived about 250 A.D., wrote a treatise introducing symbolism whose indeterminate equations are solved with rational values. Consider the problem of finding triangular numbers which are also square. We already know the formulae for both and can set them equal: $n(n+1)/2 = x^2$ or $n(n+1) = 2x^2$. $0, 1, 36, 1225, \ldots$ are solutions when $(\{n, x\} \in \{(0,0), (1,1), (8,6), (49,35), \ldots\}$. Such analysis can be quite difficult and might involve expressing square roots as continued fractions, *etc.* and sparked the early interest of many mathematicians.

12.7 Fermat's Last Theorem

Fermat considered extensions to the Pythagorean Theorem and wondered if there existed any natural numbers such that $x^n + y^n = z^n$ for $n > 2$. This became known as

Fermat's Last Theorem and was solved in the negative only in recent years. Specifically, Fermat conjectured this equation to be false. His notes are in the margin of his copy of Diophantus' *Arithmetica* where he remarked about 1637: "I have discovered a truly marvelous demonstration which this margin is too small to contain." This was, of course, written in Latin, since that is what European scientists and mathematicians communicated in until Isaac Newton's book *Optiks* was published in 1704 in the **vernacular** (language native to the region, as in English). Fermat clearly proved his theorem for $n = 4$. It is also clear that to prove it for all prime n is sufficient. Euler produced an incomplete proof for $n = 3$ in 1770 which was completed by later mathematicians. Legendre proved it for $n = 5$ in 1823. Lamé proved it for $n = 7$ in 1839. In 1850 Kummer proved it for all n's which did not divide the numerators of the Bernoulli numbers.[3] One early proof failed because prime factorization is not unique over the complex numbers. Andrew Wiles in 1993 gave a three day series of lectures where he stunned the world on the last day by completing a proof of something which implied **FLT** (**F**ermat's **L**ast **T**heorem). Although it required a little patching up over the course of the next year or so, it is now well accepted. However, at 300 pages and dependant on recent advances in mathematics, it seems doubtful Fermat ever had a proof, but his margin certainly was too small!

12.8 Perfect Cuboid

Consider a three dimensional application of Pythagorean Theorem. In a box with dimensions $3 \times 4 \times 12$, it is clear the longest (body) diagonal is 13 ($5^2 + 12^2 = 169 = 13^2$). There are 3 different lengths of diagonals on the faces:

$\sqrt{3^2 + 4^2} = 5$ $\qquad$ $\sqrt{3^2 + 12^2} = \sqrt{153}$ $\qquad$ $\sqrt{4^2 + 12^2} = \sqrt{160}$

In a **perfect cuboid** (box or rectangular parallelopiped), all seven of these numbers: three lengths, three face diagonals, and one body diagonal would be integers. This seems like a another potential EXPO project and two homework problems will give two of the three types of close encounters known. It is known that if a perfect cuboid exists, one of its sides must be at least 100 billion. It is also known that perfect parallelopipeds[4] exist.

12.9 Fermat-Catalan Conjecture

The Fermat-Catalan Conjecture is a generalization of Fermat's Last Theorem. It asks if with x, y, and z as relatively prime integers, can the equation: $x^p + y^q = z^r$, with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ be satisfied. p, q, and r are also integers. Here are the only known solutions:

[3] $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{5}{66}, \ldots$.

[4] This older spelling seems to be falling out of favor to parallelepipeds, at least by Google.

x	y	z	p	q	r
1	2	3	7	3	2
2	7	3	5	2	4
7	13	2	3	2	9
2	17	71	7	3	2
3	11	122	5	4	2
17	76271	210663928	7	3	2
1414	2213459	65	3	2	7
9262	15312283	113	3	2	7
43	96222	30042907	8	3	2
33	1549034	15613	8	2	3

For the first row, $1^7 + 2^3 = 1 + 8 = 9 = 3^2$ with $1/7 + 1/3 + 1/2 = 41/42 < 1$.

The second row has $2^5 + 7^2 = 32 + 49 = 81 = 3^4$ with $1/5 + 1/2 + 1/4 = 19/20 < 1$.

Several students in 1997–98 attempted 25000 bonus points for finding another solution and some continued their research in 2000–01 as an EXPO projects or college research.

12.10 Goldbach's Conjecture

Christian Goldbach lived in Russia 1690–1764. His mathematical work includes what has become known as Goldbach's Conjecture which states: every even number greater than 2 can be expressed as the sum of 2 primes, not necessarily distinct. No counterexample has ever been found, but a complete proof has eluded mathematicians since 1742. However, during the summer of 2003 two groups, one Chinese, one Iranian, both claimed proof. I reject the Chinese proof out of hand. They may have proved something similar, but not Goldbach's Conjecture. They assume one is prime—elsewise, it is elegant. You be the judge of the Iranian proof.

Example: $100 = 3 + 97 = 11 + 89 = 17 + 83 = 29 + 71 = 41 + 59 = 47 + 53$.

12.11 Distance Homework

All values should be given as **exact,** which means in simplified radical form. (Remember to rationalize the denominator, if necessary.) Decimal approximations are optional, but also lend completeness, but must be **clearly identified as approximations. Each problem is worth two points.**

1. Using the Pythagorean Theorem in its three dimensional form ($a^2+b^2+c^2 = d^2$), **find** exactly and **simplify** the three face diagonals and the body diagonal of a parallelopiped (box/cuboid) with $a = 240$, $b = 44$, $c = 117$.

2. Using the Pythagorean Theorem in its three dimensional form ($a^2+b^2+c^2 = d^2$), **find** exactly and **simplify** the three face diagonals and the body diagonal of a parallelopiped (box/cuboid) with $a = 104$, $b = 153$, $c = 672$.

3. Find the exact length of the hypotenuse of an isosceles right triangle if the legs are of length 5.

4. Given the hypotenuse of an isosceles right triangle as 12, what are the exact lengths of the other two sides.

5. Given a $30°, 60°, 90°$, triangle with the hypotenuse 14, find the **exact** lengths of the other two sides.

6. Given a $30°, 60°, 90°$, triangle with the side opposite the $60°$ angle being 12, find the **exact** length of the other two sides.

7. Find the exact distance between the points $(-12, 6)$ and $(4, -6)$.

8. Find the exact distance between the two points $(3, 5)$ and $(1, -1)$.

9. Driving to Dairy Queen from the MSC, you go a 1/4 mile to the left. The road bends (90°) to the right, and you proceed on for another mile to Main street. At Main Street, you take a left and continue for another 2 miles. Dairy Queen will be on the left side of the road. If you happened to walk directly from MSC to Dairy Queen, how many miles would you save by not driving?

10. George lives 5 miles north and 2 miles east of the MSC, while Jenni lives 1 mile west and three miles south of the MSC. How far apart do they live? (Assume a flat earth!)

11. A circle is the set of points equidistant from a given point. If $(4,2)$ is the center with $(6,3)$ on the circle, prove that $(2,3)$ is also on the circle. Note: $(x-h)^2+(y-k)^2=r^2$ gives the relationship for a circle centered at (h,k) with radius r.

12. The distance from point A to $(3,2)$ is 15. Find point A. How many answers could you have?

13. Verify rows 3 through 5 of the Fermat-Catalin Conjecture table.

14. Verify that Goldbach's Conjecture is true for 58 and 74. How many different sums satisfy Goldbach's Conjecture for 58? For 74? (An example is 78: 71 + 7 = 11 + 67 = 17 + 61)

15. Use your calculator (in degrees mode or use degree symbol) to verify $\sin 15° = \frac{\sqrt{6}-\sqrt{2}}{4}$ and $\cos 15° = \frac{\sqrt{6}+\sqrt{2}}{4}$, then carefully evaluate exactly $(\frac{\sqrt{6}-\sqrt{2}}{4})^2+(\frac{\sqrt{6}+\sqrt{2}}{4})^2$.

16. Verify $\tan 15° = 2-\sqrt{3} = \frac{\frac{\sqrt{6}-\sqrt{2}}{4}}{\frac{\sqrt{6}+\sqrt{2}}{4}}$.

17. Read section 8.6 in your geometry textbook and look at problems 8.6: 11–14, 18–19, 27.

Numbers Lesson 13

Cartesians, Polynomials, Quadratics

Read Euler, read Euler, he is the master [teacher] of us all. LaPlace

This lesson develops the cartesian coordinate system, relations and functions, then discusses slope, equations of a line, quadratics, the quadratic formula, the discriminant, cubics, and higher order polynomials.

13.1 Analysis Incarnate: Euler

When the four greatest mathematicians are listed, Euler's name is the one added to the great three. Leonard Euler—pronouced **Oiler**—(1707–1783) was a Swiss mathematician and physicist, although he spent most of his life in Germany and Russia. Since he published more papers than any mathematician of his time he has been called prolific—prolific can also be applied to the fact that he fathered 13 children.

Euler's father was a friend of the Bernoulli family and Euler's genius was soon discovered by them. His course of study shifted from theology to mathematics when Johann Bernoulli intervened, telling Euler's father he would be a great mathematician. Euler followed Johann's son Daniel to St. Petersburg after son Nicolas died. Euler was barely 20 when he started working at the Imperial Russian Academy of Sciences—he had just completed his Ph.D. The Academy emphasized research and had few students and a good library. After 14 years Euler moved to Berlin. While there he wrote over 200 letters to a German princess explaining diverse areas of math and science. These were compiled into a best-seller. Frederick the Great's mother had difficulty engaging Euler in conversation to which he replied: "Madam, it is because I have just come from a country where every person who speaks is hanged."

Euler lost sight in his right eye while in Russia and his sight in his left eye deteriorated while he was in Germany, rendering him nearly blind. However, Euler had phenominal mental calculation skills and a photographic memory which allowed him to compensate so his productivity seemed barely affected. "Euler calculated without apparent effort, as men breathe." Euler later returned to St. Petersburg where he worked the last 16 years of his life.

Euler developed the field of graph theory which we will discuss further in Geometry and revolutionized several other fields, such as number theory. He standardized the use of many mathematical symbols, terminology, and notation we now take for granted, such as π, e, $i = \sqrt{-1}$, Σ, $f(x)$, *etc.* His final words were: "I die" when he died of a stroke, perhaps with a child on his lap, which is how he often worked.

13.2 Introduction

Coordinate geometry was developed by both Descartes and Fermat. Today we use cartesian coordinates extensively which are named after the former. The relationship between two sets of numbers are often represented via a graph or an equation. For example: $F = \frac{9}{5}C + 32$ relates temperature in Celsius to temperature in Fahrenheit. One variable is designated the **independent variable** (C) and the value (F) depends on it and is thus the **dependent variable**. Often, it is easy to reverse these roles: $C = \frac{5}{9}(F - 32)$. Such relationships, if plotted on a coordinate system are lines and hence termed **linear**.

13.3 Ordered Pairs, Quadrants, Relations, Functions, $f(x)$, VLT

Mathematicians often speak of forming the **cartesian product** of several items. The cartesian product is a set operation, but results in a (potentially) bigger object which is generally not a member of our universal set! One example would be the rational numbers formed as ratios of integers. This one happens to be the same size as the integers.

The cartesian coordinate system is such a cartesian product of two number lines, labelled x and y. Now instead of having points on a number line with a single number to indicate its distance from the **origin** (zero), we have points on a plane with two numbers to indicate position. The number lines divide the plane into four **quadrants** labelled I, II, III, IV counterclockwise with quadrant I having both positive x and positive y coordinates. Occasionally Arabic instead of Latin numbers are used, especially when referring to a single quadrant. The axes are not in any quadrant.

II	I
III	IV

These coordinates are called **ordered pairs** and are separated by commas and enclosed within parentheses. The first coordinate **(abscissa)** is x and is plotted horizontally. The second coordinate **(ordinate)** is y and is plotted vertically. **Warning: the notation for an open interval is identical!**

Lattice points are points in the xy-plane with integer coordinates for both x and y.

A **relation** is a set of ordered pairs.

A **function** is a relation for which there is exactly one value of the dependent variable for each value of the independent variable.

Instead of writing $y = x + 2$, functional notation is often used: $f(x) = x + 2$. This does not mean to multiply f by x. It means f is the name of the function with x as the independent variable. It gives the recipe for finding $f(x) = y$ given an x value.

The set of values of the independent variable is the **domain**.

The set of values of the dependent variable is the **range**.

The **Vertical Line Test** can be used to determine if a relation is a function as follows. Check if any vertical line ever crosses the relation more than once. If it does, the relation has failed the vertical line test and is not a function.

13.4 Slope, Line Equations

About half of calculus is concerned with finding the slope of any function anywhere. Slope is thus an important concept but should already be familiar.

slope $= m = \text{rise}/\text{run} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\Delta y}{\Delta x} = \dfrac{dy}{dx}$.

Parallel lines have equal slopes.
Perpendicular lines have slopes which are negative reciprocals.

Note: modern books tend to use an inclusive definition of parallel which allows a line to be parallel to itself. Others exclude this.

This should be well studied in Algebra, so only a quick review is presented in today's activity. In summary, if $y = mx+b$, then m is the slope and b is the y-intercept (*i.e.*, the value of y when $x = 0$). Often linear equations are written with integer coefficients in either **standard** ($Ax + By = C$) or **general** ($Ax + By - C = 0$) **form**. Such relationships must be converted into **slope-intercept form** ($y = mx + b$) for easy use on the graphing calculator. In today's activity $-10x + y = -5$ ($10x - y = 5$) and $y = 5$ are encountered. Such **systems of equations** are either **inconsistent** (parallel lines, so have no points in common), **dependent** (coincident lines (same

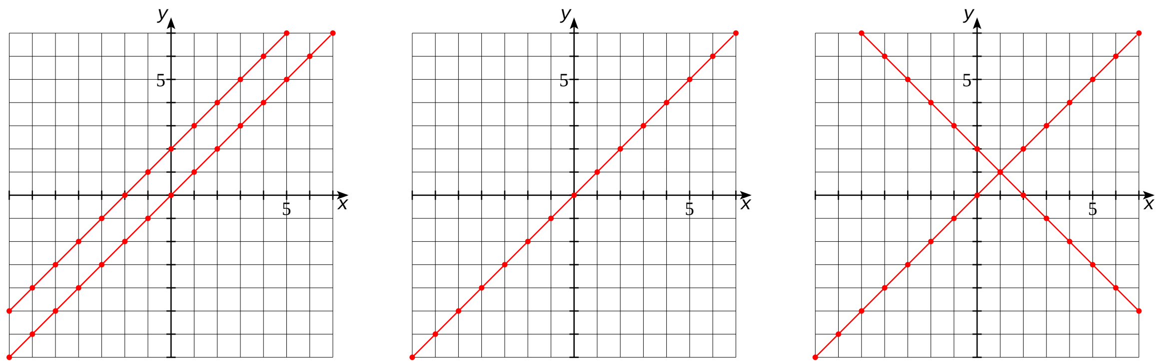


Figure 13.1: Systems of equations can be inconsistent (left with $y = x$ and $y = x+2$), dependent (middle with $y = x$ and $2y = 2x$), or independent (right with $y = x$ and $y = -x + 2$).

slope and y-intercept), so all points are in common), or **independent** (slopes are different). See Figure 13.1. One other form of an equation for a line is called the **point-slope form** and is as follows: $y - y_1 = m(x - x_1)$. The slope, m, is as defined above, x and y are our variables, and (x_1, y_1) is a point on the line.

13.5 Special Slopes

It is important to understand the difference between **positive**, **negative**, **zero**, and **undefined slopes**, and that is also covered in today's activity. In summary, if the slope is positive, y increases as x increases, and the function runs "uphill" (going left to right). If the slope is negative, y decreases as x increases and the function runs downhill. If the slope is zero, y does not change, thus is constant—a horizontal line. Vertical lines are problematic in that there is no change in x. Thus our formula is undefined due to division by zero. Some will term this condition **infinite slope**, but be aware that we can't tell if it is positive or negative infinity! Hence the rather confusing term **no slope** is also in common usage for this situation.

13.6 Polynomials

Polynomials are algebraic expressions involving only the operations of addition, subtraction, and multiplication $(+, -, \times)$ of variables. The coefficients should be rational or perhaps real.

Polynomials involve no nonalgebraic operations (such as absolute value) and no operations under which the set of real numbers is not closed, such as $\div$ or square root.

An **expression** is a collection of variables and constants connected by operation symbols ($+, -, \times, \div$, *etc.*) which stands for a number.

A **term** is a part of an expression which is added or subtracted.

Quadratic functions are polynomials with degree two and will be explored below.

The **degree** of a polynomial is the maximum number of variables which are factors in any one term.

Polynomials (poly- means **many**) are named based on how many terms they have **and** by their degree.

Monomials have one term.
Binomials have two terms.
Trinomials have three terms.

Linear functions are a special class of polynomials with degree one. A constant function has degree zero.

If only one variable is present, such as x, we have a **polynomial in** x. The coefficient of the term with highest degree is called the **leading coefficient**. There may also be a **constant coefficient** which has no x multiplier.

13.7 Quadratic Functions

The general equation for a quadratic function is $y = ax^2 + bx + c$, where a, b, and c are constants, and $a \neq 0$. (If $a = 0$, then the function is linear.)

Learn the **Quadratic Formula** (its derivation is given below): $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$ax^2 + bx + c = 0$	Given: the general quadratic equation
$ax^2 + bx = -c$	Move constant to other side, by subtracting c from both sides.
$\frac{ax^2}{a} + \frac{bx}{a} = \frac{-c}{a}$	Remove coefficients from quadratic term (x^2) by dividing everything by the coefficient.
$x^2 + bx/a + (b/2a)^2 = -c/a + (b/2a)^2$	To have perfect square trinomial (that's why method is called Completing the Square), need to take half of "b", square it, and add that to both sides.
$(x + b/2a)(x + b/2a) = -c/a + (b/2a)^2$	Factor left side since it is now a perfect square.
$(x + b/2a)^2 = -c/a + (b^2/4a^2)$	Rewrote in exponential form ($x \times x = x^2$).
$(x + b/2a)^2 = \frac{-4ac}{4a^2} + \frac{b^2}{4a^2}$	On the right side, rewrote fractions to have common denominator, $4a^2$.
$x + b/2a = \frac{\pm\sqrt{-4ac+b^2}}{2a}$	Took square root of both sides (As you do to one side, do to the other.) When adding fractions with a common denominator, add the numerators.
$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$	Isolate the variable by subtracting $b/2a$ from both sides.

The shape of the graph of a quadratic equation is called a **parabola**. On both sides of the **vertex** (the maximum or minimum point on the graph), the graph of the equation either increases or decreases. The vertex lies on the **axis of symmetry**. Thus the graph on one side of the line (axis) of symmetry is a reflection of the graph on the other side. Several examples of parabolas are explored in today's activity.

Where the graph crosses the x-axis are points called **x-intercepts** where $y = 0$. The general equation then degenerates into $ax^2 + bx + c = 0$. To solve for x, the quadratic formula method must be mastered. It involved fractions and radicals. **Quadratic Relations** will be explored in Algebra II, Precalculus, and Calculus BC. They will allow the full nature of **conic sections** to be explored.

To obtain the solution to a quadratic equation, Completing the Square is sometimes used. Using the completing-the-square method, as outlined above in the derivation of the quadratic formula, on the general equation ($ax^2 + bx + c = 0$) will find the solutions to any equation.

13.8 Discriminant

If $ax^2 + bx + c = 0$, then the quantity $D = b^2 - 4ac$ is called the **discriminant.**

Gauss's **Fundamental Theorem of Algebra** states that the number of solutions to any equation cannot exceed its degree. In fact, if we carefully count repeated (see Activity 12) and complex roots (see Numbers Lesson 16), we will find equality. So, a quadratic equation may have up to two solutions. To determine quickly how many and what type of solutions a quadratic equation has, analyze the discriminant.

Given: $ax^2 + bx + c = 0$, where a, b, and c are real numbers.		
If $b^2 - 4ac$	< 0	The equation has no real-number solutions. The solutions, involving non-real complex numbers, will be discussed in Numbers Lesson 16.
If $b^2 - 4ac$	> 0	The equation has two different real-number solutions. If D is a perfect [rational] square, the solutions are rational.
If $b^2 - 4ac$	$= 0$	Then the equation has a repeated real-number solution with the vertex on the x-axis. If a and b are rational, then the solution will also be rational.

An example is $x^2-6x+8 = 0$ where $a = 1$, $b = -6$, and $c = 8$. So the discriminant becomes $(-6)^2 - 4(1)(8) = 36 - 32 = 4$. Since 4 is a positive number, the equation will yield two real-number solutions. These answers are $(6 + 2)/2$ and $(6 - 2)/2$, which reduce to 4 and 2. These are related to the original equations as follows: $x^2 - 6x + 8 = (x - 4)(x - 2) = 0$.

13.9 Solutions, Roots, Zeroes, and x-intercepts

The four terms **solutions**, **roots**, **zeroes**, and **x-intercepts** are often used somewhat interchangeably to refer to the values of x where an equation is zero.

13.10 Cubic, Quartic, Quintic

Polynomials with degree three are referred to as **cubic functions**. Degree four polynomials are **quartic functions** and degree five polynomials are **quintic functions**.

There are ways to solve cubic functions and quartic functions, but the general quintic function $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$ is not solveable algebraically—only numerical approximation can be obtained. Polynomials in x with only even or odd exponents are termed **even** or **odd**. This terminology is carried over to other graphs which have similar symmetry when graphed. See Figures 13.2 and 13.3.

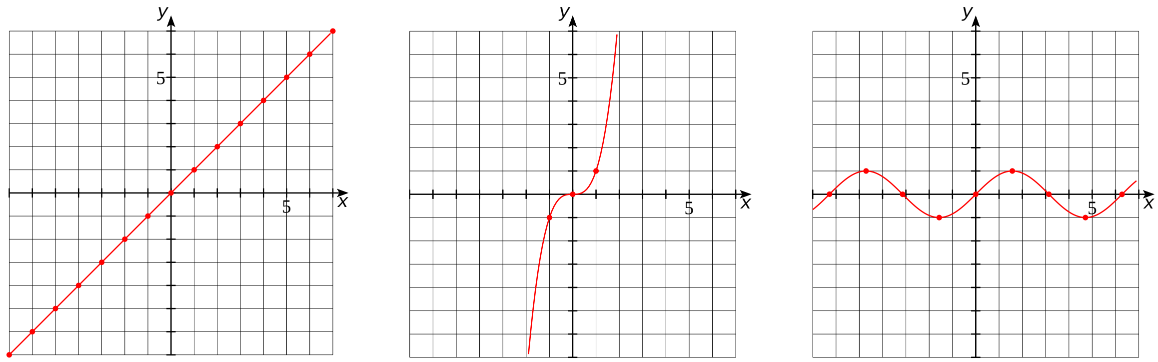


Figure 13.2: Odd Functions: $y = x$ (left), $y = x^3$ (middle), and $y = \sin x$ (right), where $f(-x) = -f(x)$.

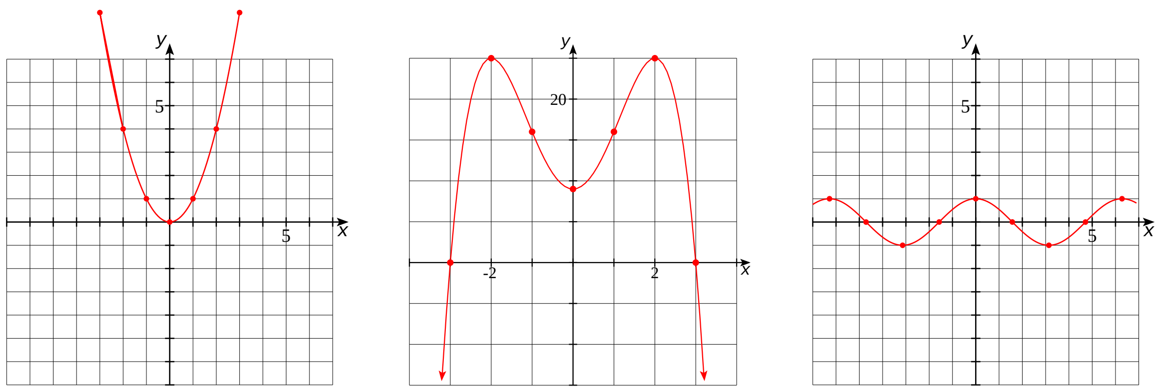


Figure 13.3: Even Functions: $y = x^2$ (left), $y = -(x^2 - 9)(x^2 + 1)$ (middle), and $y = \cos x$ (right), where $f(-x) = f(x)$.

For example, the sine function is termed odd because $\sin(-x) = -\sin x$, whereas the cosine function is termed even because $\cos(-x) = \cos x$, similar to what happens with polynomials with only even or odd degree terms. The even functions are symmetric about the y-axis, but the odd functions are symmetric about the origin.

13.11 Functions Homework

Each problem is worth two points.

1. Aunt Ethel hands you \$15 in quarters (q) and dimes (d). Name five ordered pairs (q, d) representing the change she might have given you. **Graph** the points. What relation do you observe?

2. What are the slopes of the line containing points $(0, 2)$ and $(9, 5)$ and the line with points $(-1, 4)$ and $(5, 8)$? Which line is steeper?

3. Prove that "If two lines are parallel to the same line, then they are parallel to each other."

4. If the slope of a line is $\frac{-3}{4}$, what is the slope of a perpendicular line to it?

For problems 5–8, classify the following lines as vertical, horizontal, or **oblique** (neither):

5. $x + y = 2$.

6. $2x = 6$.

7. $3x - 2y = 1$.

8. $y = 17 - 5$.

9. **Graph:** $y = 3x + 2$.

10. **Graph:** $x + 4y = 4$.

11. Determine if the following system of equations is inconsistent, independent, or dependent:

$$\begin{aligned} 2x - 3y &= 5 \\ 10x - 15y &= 25. \end{aligned}$$

12. Determine if the following system of equations is inconsistent, independent, or dependent:

$$\begin{aligned} 6x + 4y &= 3 \\ x - 1.5y &= 4. \end{aligned}$$

13. Find a line perpendicular to the given line: $4x - y = 3$.

14. **Graph** the equation $y = x^2 - 3$. Is it a relation or a function?

15. **Graph** the equation $x^2 + y^2 = 4$. Is it a relation or a function? (If doing by calculator, solve for y. Enter into calculator both branches for y due to $\pm$ the square root.)

16. **Graph** the function $y = x^2 + 5x + 6$. Find the domain and range.

17. **Graph** the function $y = x^2 - 4x + 4$. Find the domain and range.

18. Solve the equation for x **exactly**: $5x^2 + 8x - 6 = 3$.

19. Determine if the equation has real solutions. $4x^2 - 13x + 11 = 0$.

20. Solve the equation, $y = x^2 - 4x + 5$ **exactly**, when $y = 0$. What does this infer about the graph of the function?

21. Read sections 3.6 and 3.8 in your geometry textbook and do problem 10 in both.

Numbers Lesson 14

It's Been Real

Wir müssen wissen, wir werden wissen.[1] David Hilbert

In this lesson we will extend our understanding of numbers beyond the rational to the reals—*i.e.* all the numbers on the real number line. We will state various facts about the irrationals and reals, discuss continuity and denseness, prove the reals to be nondenumerable, present the field axioms used with the real numbers, including the Peano Axioms of Arithmetic, and Trichotomy. We discuss orders of infinity and some of Gödel's work. We present the axioms of set theory, and close with a section on paradoxes. Some of this makes for heavy reading and is here more for reference than mastry at this time.

14.1 Father of [in]completeness: Kurt Gödel

Kurt Gödel (1906–1978) is one of the two most important logicians, the other being Alfred Tarski (1902–1983). Kurt Gödel is generally considered an Austrian-American mathematician although he was born in an area which is now in the Czech Republic. He became Czech upon the political organization at the end of World War I, and became a German citizen when Germany took over Austrian (Anschluss) in 1938. Gödel and his wife left Vienna in 1940 and travelled via the trans-Siberian railway, Japan, and California to the Institute of Advanced Studies in Princeton, NJ. He had visited Einstein and others there several years earlier and even spent a year at Notre Dame.

By 1931 Gödel unveiled Gödel's incompleteness theorem for which he is best known. It proved that for any computable axiomatic system strong enough to describe arithmetic on the natural numbers: 1) if it was consistent, then it was incomplete; 2) the consistency of the axioms could not be proved within the system. This ended a

[1] We must know, we shall know.

half century of attempts epitomized by Hilbert, Whitehead and Russell, of finding a set of axioms sufficient for all mathematics.

Before coming permanently to the US, Gödel was able to show that the Axiom of Choice (AC) and the Generalized Continuum Hypothesis (GCH) were true in a set theory model (using the Zermelo-Frankel axioms or ZF) known as the constructible universe and thus consistent with the standard axioms of set theory. During the 1960's Paul Cohen developed a model in which they were false thus showing their independence. More on these below

14.2 Reals

There are numbers on the number line which are not rational.

We already showed that the $\sqrt{2}$ was irrational. We also stated that the rationals were **dense**—between each rational number was another rational number. However, apparently they are not **continuous** or **complete**. Somehow if we only had rational numbers on our number line, we would skip over the $\sqrt{2}$ even though any decimal approximation, such as $1.414, 1.4142, \cdots$, is on our number line!

The **Real Numbers** are all the numbers on the number line.

Physicists like to say that they work with continuous functions with continuous derivatives (slopes), whereas mathematicians spend a lot of time worrying about whether or not a function or its derivatives are continuous. You will explore this concept further in Algebra II and Calculus. Suffice it to say now that if you can plot the function without picking up your pencil, it is continuous. A number line is such a plot.

Real Numbers are either rational or irrational.
All rational and all irrational numbers are real numbers.

The rational and irrational numbers are disjoints sets which together make up the real numbers.

The symbol $\Re$, $\mathcal{R}$, or $\mathbf{R}$ denotes the set of real numbers.

$$\mathcal{N} \subset \mathcal{Z} \subset \mathcal{Q} \subset \mathcal{R} \qquad \text{or} \qquad \mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q} \subset \mathbf{R}$$

John Derbyshire in *Prime Obsession*, page 170, offers the mnemonic: Nine Zulu Queens Rule China to help remember how these nested Russian dolls are arranged.

The real numbers are **nondenumerable** (uncountable).

Proof by contradition:
Assume that the real numbers are denumerable (meaning, they have one-to-one correspondence to natural numbers). Then there exists a pairing of each number such

that neither set has any elements left over. The following notation indicates one such pairing where the a's, b's, c's, *etc.* represent digits and the subscripts indicate the location to the right of the decimal point: $1 \leftrightarrow 0.a_1a_2a_3\cdots$, $2 \leftrightarrow 0.b_1b_2b_3\cdots$, $3 \leftrightarrow 0.c_1c_2c_3\cdots$, *etc.* But we will now show that there is at least one real number which is not included in this pairing. Let $N = 0.n_1n_2n_3\cdots$, where the n's represent any digits such that: n_1 is not equal to a_1, n_2 is not equal to b_2, n_3 is not equal to c_3, *etc.* Thus N is a real number and is different from each of the real numbers in the one-to-one correspondence. Thus the set of real numbers is non-denumerable. This proof goes back to Georg Cantor in 1874.

14.3 The Field Axioms

We introduced the group axioms in Number Lesson 8. Another interesting mathematical object is a **ring**. They have two operators usually called addition (+) and multiplication (× or • or just **juxtapositioned** (from Latin: to be placed side by side)). Since × and x can so easily be confused, • is often preferred. A ring is an **abelian** group under addition, where abelian means it is commutative (see the axiom below), and comes from a famous Norwegian mathematician named Niels Henrik Abel (1802–1829). (Abel is generally pronounced with a long e sound and accented second syllable.) A ring must also be closed under multiplication, and must also be associative (for an associative ring). There is also an axiom to interrelate addition and multiplication (see the distributive property below). The rings of interest to us have a **unit element** which will serve as our multiplicative identity (1), and are commutative under multiplication. A field is just another mathematical object with more structure than a ring.

> If the elements different from 0 in a commutative ring with unit element form an abelean group under multiplication, the ring is called a **field**.

Zero must be excluded because it does not have a multiplicitive inverse—division by zero is not allowed. The only fields we will be concerned with are the binaries (0,1), the rational numbers, the real numbers, and in Numbers Lesson 16, the complex numbers.

The eleven field axioms are listed below and are true for any real numbers, represented below by x, y, and z.

> **Closure under addition**: real numbers are *closed* under addition.

That is, adding any pair of real numbers will result in a **unique** real number. $1 + 1 = 2$. Always. This also means we stay inside the set.

> **Closure under multiplication**: real numbers are *closed* under multiplication.

Multiplying any real number pair together will result in a **unique** real number.

$2 \times 2 = 4$ and never 5.

Additive Commutativity: $x + y = y + x$.

Order does not matter. You can add a column of numbers from the top or from the bottom.

Multiplicative Commutativity: $x \bullet y = y \bullet x$.

The root word commute is commonly used to describe exchanging places, like going forth and back between home and work.

Additive Associativity: $(x + y) + z = x + (y + z)$.

Multiplicative Associativity: $(xy)z = x(yz)$.

Distributivity: Multiplication distributes over addition. $x(y + z) = xy + xz$.

Additive Identity Element: The additive identity is a *unique* element, which can be added to any element without altering it. The additive identity is zero (0). $x + 0 = x$.

We have both a left and right additive identity element and they are the same: $x + 0 = x = 0 + x$.

Multiplicative Identity Element: The multiplicative identity is unique; it is one (1). $x \bullet 1 = x$.

We also have both a left and right multiplicative identity element and they are the same: $x \bullet 1 = x = 1 \bullet x$.

Additive Inverses: For every real number there exists a *unique* inverse, such that when added together, the result is the additive identity (0). The additive inverse is the opposite (negative) of the given real number, $x + (-x) = 0$.

Multiplicative Inverses: For every real number not equal to zero there exists a *unique* inverse, such that when multiplied together, the result is the multiplicative identity. $x \bullet x^{-1} = 1$.

x^{-1} is a general designation for an inverse, but here denotes the multiplicative inverse or reciprocal $(1/x)$.

14.4 Reflexive, Symmetric, Transitive, Closure, Trichotomy

The three axioms of Reflexive, Symmetric, and Transitive, can be used to define equality. In fact, these three are often added to the five Peano axioms given in Lesson 2 to form **Peano Arithmetic.** In this situation they are applied to the natural numbers only. One additional axiom is needed, that of closure for equality, which is given below.

In addition to the field axioms, real numbers satisfy additional important axioms or properties.

Reflexive Property: If x is a real number, then $x = x$.

Operations which are reflexive look the same in a mirror. This axiom establishes that a variable stands for the same number wherever it appears in an expression. Order is not reflexive: $5 < 5$ is a counterexample.

Symmetry: If $x = y$, then $y = x$.

Notice that symmetry is true for only the equal ("=") sign. Order relationships, such as $<$ and $>$, cannot have the numbers rearranged without changing the meaning. For example, $4 < 5$ is not the same as $5 < 4$.

Transitivity: If $x = y$ and $y = z$, then $x = z$.
If $x < y$ and $y < z$, then $x < z$.
If $x > y$ and $y > z$, then $x > z$.

The prefix *trans-* means **across** like rapid transit quickly takes you across a city. An easy way to remember which of these three properties is which is to note that the initial letters **RST** are in alphabetic order and corresponds to **123** or the number of variables which appear in the description!

Closure: For all a and b, if a is a natural number and $a = b$, then b is also a natural number.

That is, the natural numbers are closed under equality. We stated it for natural numbers to complete the list of nine Peano axioms, but it can also be accepted for real numbers.

Trichotomy: If x and y are two real numbers, then *exactly one* of the following must be true: $y < x$, $y > x$, or $y = x$.

Trichotomy means to section or cut into three pieces. Please note it is three pieces not two because the reals are continuous (not just dense). You will hit a number wherever you cut the real number line.

14.5 Higher Orders of Infinity, $\aleph_n$

George Cantor introduced transfinite numbers back in the 1870's as a way to deal with the fact that not all infinite sets are equivalent. The cardinality of the integers, rational numbers, even algebraic numbers is designated the first order of infinity and assigned the name aleph null ($\aleph_0$) where aleph ($\aleph$) is the first Hebrew letter. However, the cardinality of the real numbers or such important subsets as the transcendentals or irrationals is beyond that of a countable infinity. This cardinality became known as the cardinality of the continuum and was designated by c. By forming power sets (the set of all subsets of a given set), Cantor was able to form higher order infinities. These became known as $\aleph_0, \aleph_1, \aleph_2, \cdots$, where $2^{\aleph_0} = \aleph_1$ Cantor believed this first aleph ($\aleph_1$) was the cardinality of the continuum and was sometimes able and sometimes not able to prove it. This may well have been a contributing factor to his mental instabilities. This hypothesis ($2^{\aleph_0} = \aleph_1$) became known as the **Continuum Hypothesis** (CH).[2] This power set relationship was later generalized to apply to any successive pair of alephs and became known as the generalized continuum hypothesis. Only much later was it shown that CH is independent of the usual axioms of set theory and was thus unproveable (Kurt Gödel, 1937 and Paul Cohen, 1963). The method used by Cohen became known as **forcing**.

While we are on the topic, another axiom, the **axiom of choice** (AC) suffered a similar fate, being proved independent of the rest of mathematics (Gödel, 1940 and Cohen, 1963). However, unlike CH, it is still routinely, but not universally, used in the development of mathematics.[3] One last related topic is Gödel's **Incompleteness Theorem**, 1931, which showed that there were things within any formal system which were neither provable nor not provable. These recent developments make one question the very merits of establishing a rigorous foundation for mathematics!

14.6 The Axioms of Set Theory

Following are the axioms of set theory generally used in mathematics. They were designed by Ernst Zermelo, *et al* at the beginning of the 20$^{\text{th}}$ century. This minimal set of assumptions leads to a consistent body of mathematical knowledge, including the natural, real, and complex numbers along with their properties and arithmetic. Along with other axioms, the areas of geometry, algebra, topology, *etc.* can also be formed. Georg Cantor developed set theory but implicitly assumed many of these.

- **Existence:** There exists at least one set. (The empty set can be chosen. The set containing the empty set would then be constructed $\cdots$.)

[2] `http://www.ii.com/math/ch/`

[3] `http://www.cs.unb.ca/~alopez-o/math-faq/mathtext/node35.html`

- **Extension:** Two sets are equal iff they have the same elements.
- **Specification:** To every set A and every condition $S(x)$ there corresponds a set B whose elements are exactly those elements x of A for which $S(x)$ holds. This axiom leads to Russell's paradox.
- **Pairing:** For any two sets there exists a set to which they both belong.
- **Unions:** For every collection of sets there exists a set that contains all the elements that belong to at least one of the sets in the collection.
- **Powers:** For each set there exists a collection of sets that contains among its elements all the subsets of the given set.
- **Infinity:** There exists a set containing 0 and containing the successor of each of its elements.
- **Choice:** For every set A there is a choice function, f, such that for any non-empty subset B of A, $f(B)$ is a member of B.

14.7 Surreal Numbers

John Conway invented surreal numbers in recent years. These numbers have multiple infinities and many other unusual but useful properties. Donald Knuth wrote a novellete to help explain these numbers even before the technical paper was published.

14.8 Continuity

Our macroscopic existence means that most of our physical observations are continuous. Thus most physical phenomina is modelled by continuous functions with continuous derivatives (slopes). Some cutting edge models attempting to unify gravity with quantum mechanics while retaining general relativity (as in loop quantum gravity, unlike string or M-theory) treat space as quantized. However, the mathematical treatment of functions is riddled with concerns about continuity. Discontinuities fall into two catagories: **removable** and **nonremovable**. We stated before that continuous functions can be drawn without having to lift your pencil from the paper. For removable discontinuities one must only avoid an occasional point whereas nonremovable discontinuities involve moving your pencil up or down. The function x/x would have a removeable discontinuity at $x = 0$, whereas $|x|/x$ would have a nonremoveable discontinuity. The definition of continuity is wrapped up with the concept of **limit** and will not be discussed further here.

14.9 Paradoxes

We already encountered various paradoxes in Numbers Lesson 1 (Barber, Russell's) and Lesson 6 (Liar's). Several paradoxes dating back to the ancients are presented below. Zeno's name is often associated with these and other equivalent ones which show that motion is only an illusion. Even in ancient times these were considered absurb, but it took a modern understanding of infinity, infintesimals, and convergent infinite series to dispel most (not all!) doubt.

14.9.1 Paradox: Dichotomy

You cannot even start.

"That which is in locomotion must arrive at the half-way stage before it arrives at the goal."—Aristotle.

14.9.2 Paradox: Archilles and the Tortoise

You can never catch up.

Aristotle rendered this paradox as follows: "In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead."

14.9.3 Paradox: Arrow

You cannot even move.

"If everything when it occupies an equal space is at rest, and if that which is in locomotion is always occupying such a space at any moment, the flying arrow is therefore motionless."

This paradox, instead of dividing up space like the prior two, divides time.

14.10 Real Homework

Each problem is worth two points.

1. Name the axiom used: $10 + 13 + 17 + 23 = 10 + 17 + 13 + 23$.

2. Name the axiom used: $14 \bullet ((17 + 52) + 30) = 14 \bullet (17 + (52 + 30))$.

3. Name the axiom used: $7 \times 11 \times 13 = 11 \times 7 \times 13$.

4. Name the axiom used: $\sqrt{(7 \times 11) \times 13} = \sqrt{7 \times (11 \times 13)}$.

5. Name the axiom**S** used: $x + 0 = x$ **always**.

6. Show by counterexample that subtraction is not commutative.

7. Show by counterexample that subtraction is not associative.

8. Show by counterexample that negative numbers are not closed under multiplication.

9. Show by counterexample that there is no Symmetric Property of greater than ("$>$").

10. Show by counterexample that not equal ("$\neq$") is not transitive.

11. Is the relationship of "Alexis is a sister of Tom" symmetric? Show by example why or why not.

For problems 12–15, which **field axioms** do the following sets of numbers fail? An example is irrational numbers failing for closure under multiplication since $\sqrt{2}\sqrt{2} = 2$, which is rational.

12. Natural numbers ($\mathcal{N}$).

13. The integers ($\mathcal{Z}$).

14. The rational numbers ($\mathcal{Q}$).

15. The binary digits $\{0,1\}$ with **and** as the multiplication type operator ($\times$) and **eor** (or modulo 2 addition) as the addition type operator ($+$), the only difference is "1+1=0").

16. Consider again the set $\{0,1\}$ with **and** and **or** as operations. Does **and** distribute over **or** as well as *vice versa*? Fill in the table to prove or disprove these distribution rules.

p	q	r	$p \bullet (q \vee r)$	$(p \bullet q) \vee (p \bullet r)$	$p \vee (q \bullet r)$	$(p \vee q) \bullet (p \vee r)$
0	0	0				
0	0	1				
0	1	0				
0	1	1				
1	0	0				
1	0	1				
1	1	0				
1	1	1				

17. Read section 3.4 in your geometry textbook. Do problems 3.4: 4 and 16.

Numbers Lesson 15

Transcendental Meditations

> *Who of us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science and at the secrets of its developments during future centuries?* David Hilbert

In this lesson we will discuss numbers which are not solutions to polynomial type equations and are thus termed nonalgebraic or transcendental. After introducing the Dedekind cut as a way to define real numbers, we discuss nonalgebraic numbers such as π and e. Next we discuss the related problem of geometric constructions which the ancients found impossible and which have since been proven impossible. We end the lesson by noting how many more nonalgebraic numbers there are than algebraic numbers.

15.1 The Father of Logarithms: John Napier

John Napier (1550–1617) was born, lived, and died in Scotland. He is remembered as both a mathematician and physicist and is best remembered for inventing logarithms and Napier's bones. Logarithms made hand-calculations involving multiplication and division much easier and quicker by turning them into addition and subtraction. This paved the way for many scientific advances, such as the calculation of Mars' orbit by Kepler.

Napier was also trained in theology but encouraged people to think he dabbled in black arts. Many stories have been preserved about his exploits. We will relate two here.

Napier and his servants discovered the neighbor's pigeons were helping themselves to his grain. Napier warned his neighbor he would keep any pigeons found on his property. The next day Napier was observed scooping up pigeons into sacks—he had spiked peas with brandy which they had eaten, eaten enough to be unable to fly!

Napier suspected one of his servants was stealing from him. He took a black rooster, coated it with charcoal, and put it in a dark room. All the servants were instructed to enter the room and pet the rooster. The guilty party was soon identified as the one with clean hands—every one else had done as instructed!

Napier was the Lord for his manor and thus had a very practical interest in such things as fertilizer and the water level in coal mines. Napier's favorite book was his book on the book of Revelation.

Henry Briggs (1561–1631) was so impressed with Napiers invention of logarithms that he resolved to meet their inventor in person: "where almost one quarter of an hour was spent, each beholding other with admiration, before one word was spoke. At last Briggs said: 'My lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help in astronomy, *viz.* the logarithms; but, my lord, being by you found out, I wonder nobody found it out before, when now known it is so easy.'" (*viz.* is an abbreviation for **videlicet**, Latin for **namely**.) Briggs proposed two modifications which resulted in our base 10 or common logarithms. Briggs published tables accurate to 14 decimal places for all integers 1 to 20,000 and from 90,000 to 100,000 in 1624 in *Arithmetica logarithmica* with the gap filled in by someone else by 1628. This work remained the basis for all subsequent log tables up until 1924 when a 20 decimal place table was begun to celebrate 300 years of logarithms. About 1620, the **slide rule** was also invented which is laid out on a logarithmic scale and thus by adding and subtracting distances, multiplication and division are performed.

15.2 Reals Defined Via Dedekind Cut

Transcendental numbers have a long history, dating back to the ancient Greeks, even though they were not named or truly recognized until much later. As mentioned earlier, the ancient Pythagorean school discovered the existence of irrational numbers, with $\sqrt{2}$ being the prototypical example as the diagonal of a **unit square**. They then regarded it as a numberless magnitude—distinct from an arithmetic number—a concept which remained an essential element of Greek mathematics. Soon other irrational numbers were found: the square root of every prime number, then the square root of most composite numbers. Irrational numbers, or **incommensurables** were well studied by the time Euclid wrote his *Elements*. However, it was not until 1872 when Richard Dedekind (1831–1916) published his *Continuity and Irrational Numbers* that a satisfactory theory developing such numbers was given, one devoid of geometric considerations. His **Dedekind Cut** was an essential part of that development and goes beyond what we can cover here. An alternative approach using a Least Upper Bound Axiom is also beyond our scope.

15.3 The Story of π

The concept of π was invented to simplify calculations involving circles. The Rhind Papyrus, an Egyptian text from 1650 B.C. contains a statement relating as equals, the areas of a circle and a square whose side is 8/9 the circle's diameter. This value for π of $256/81 \approx 3.16049\cdots$ is a much better value than the one recorded about 700 years later and given biblically in I Kings 7:23. "And he made a molten sea, ten cubits from one brim to the other...and a line of thirty cubits did compass it round about." These both recognize the need to relate the **diameter** or **radius** of a circle to its **area** or **circumference**. Euler was the one to attached the symbol π to the concept.

π is in fact defined as the ratio of a circle's circumference (C) to its diameter (d): $\pi = C/d$.

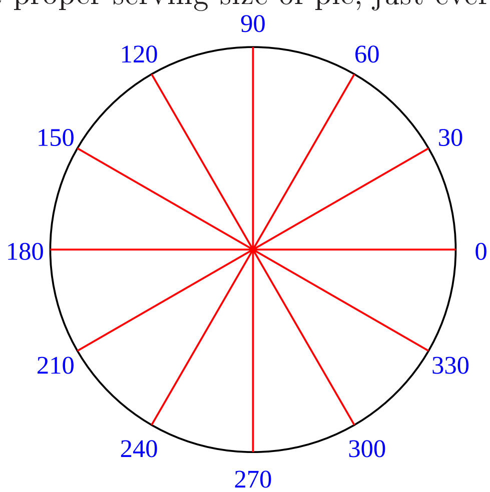
This gives the formulae: $C = \pi d = 2\pi r$, where r is the radius.
The area formula is similar: $A = \pi r^2$.

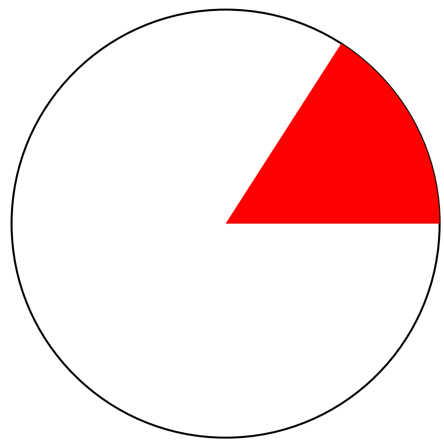
Archimedes first proposed a method of obtaining the value of π to any desired accuracy by calculating the **perimeter** of **inscribed** and **circumscribed** polygons. By increasing (usually by doubling) the number of sides, the accuracy is increased—the true value of π is squeezed between these two values. Using his crude numerical representation, Archimedes was able, by using polygons of 96 sides (bisecting the sides of a hexagon 4 times), to determine: $3\frac{10}{71} < \pi < 3\frac{10}{70}$ or $3.140845\cdots < \pi < 3.142857\cdots$ or $\pi \approx 3.1418$. Over the centuries this value was highly refined until hundreds of decimal places were known before the invention of computers and now trillions of digits are known. An interesting challenge has been memorizing these random digits and the current record is about 83,000 digits, requiring many hours to recite. (The author had 750 digits well memorized and almost had one thousand at age 16 when he thought the record was only a thousand. He has since forgotten most all but the initial 50 which he memorized at age 11.)

$\pi = 3.14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288\ 41971\ 69399\ 37510\ 58209\ 74944\cdots$

Historically, the value $\pi \approx 22/7$ was used and is within 0.04% of the true value. Such a rational approximation was useful before calculators were invented and older geometry books have many problems which were done very easily using this value. The curious value $\pi \approx 355/113$ can easily be remembered because each of the first three odd number is repeated once and is even closer to the true value. $\pi^2 \approx 9.8696\cdots$ is surprisingly close to 10, our preferred base. When students omit parentheses in denominators on their calculators, their answers are often about an order of magnitude off for this reason.

Extending the above definition of π results in its most common usage: angle measurement. The radius of a circle seems like a useful unit to measure **arc lengths** or **angles**. Note how the circumference of a **unit circle** (one with $r = 1$) is $2\pi \approx 6.28318\cdots$. An arc the length of one radius is known as a **radian** and there are 2π radians in one **revolution** or full circle (360°). Thus π radians are 180° and 1 radian is $57.2957795\cdots°$ or $57°17'44.806\cdots''$. The conversion of radians to degrees is done by multiplying the radians by $180°/\pi$. To convert degrees to radians, multiply the degrees by $\pi/180°$. The circle below is partitioned into standard angle measure in degrees. It is important to know these.[1] Mathematicians like to think of a radian as the proper serving size of pie, just ever so slightly less than 1/6.





Pi shows up in some unusual places, especially in probability. Buffon's needle is one of the originals but there are many variations, such as `http://www.wikihow.com/Calculate-Pi-by-Throwing-Frozen-Hot-Dogs` which is fairly self-explanitory.

15.4 The Story of e

Another important number to mathematics has a much shorter history than π.

Logarithm means *ratio number*. Although Napier's usage was slightly different, the modern definition is:

$$\boxed{\log_b a = c \text{ if and only if } b^c = a, \qquad b > 0, \text{ and } b \neq 1.}$$

We thus see that exponentiation (exp) is an **inverse operation** of logarithm (log). Inverse operations have already figured prominently as in subtraction is the inverse operation of addition and division is the inverse operation of multiplication. Another important one is square root as the inverse operation of squaring. **Inverse functions** can have important restrictions which differ from the original function!

Logs can be defined to any positive base (except 1), but two bases have become most prevalent: $b = 10$ (for **common logs**), and $b = e$ (for **natural logs**). Both

[1] Knowing the radian values is also important but haven't been put on this graphic yet.

appear on most calculators. The base is often omitted and high school and chemistry students can usually assume $\log x = \log_{10} x$. However, in college math and physics, $\log x = \log_e x$.

$$\boxed{\log_e x = 2.30258\cdots \log_{10} x \qquad \text{where } 2.30258\cdots = \log_e 10 = \frac{1}{\log_{10} e}}$$

$\ln x$ is fairly commonly used for natural logs (and now rarely looks like $1n$). Napier's base was $b = .9999999 = 1 - 10^{-7}$, which may be only slightly more understandable when you realize that decimal fractions were not yet widely used—Napier actually being the one to invent and popularize the decimal point! In making this choice, Napier came within epsilon (a hair's breadth) of discovering the limit of $(1 - 1/n)^n$ as n tends to infinity, which is merely the reciprocal of $(1 + 1/n)^n$ as n tends to infinity.

$$\boxed{\lim_{n\to\infty}(1+\tfrac{1}{n})^n = e.}$$

This latter value is:
$e = 2.71828\ 18284\ 59045\ 23536\ 02874\ 71352\ 66249\ 77572\ 47093\ 69995\cdots$

Logarithms were quickly adopted by scientists all over the world because they simplified calculations by turning multiplication and division into table look-ups, addition and subtraction, and then another table look-up to find the **antilog**. Like we saw in scientific notation, the decimal part of a logarithm is often called the mantissa. The integer portion is called the characteristic.

15.5 Geometric Constructions

The transcendental story really began with the restrictions the ancient Greeks (Plato) put on their **Geometric Constructions**. The only tools allowed were an **unmarked straight-edge** and **a pair of compasses**. (Most sources specify a compass, but some constructions require two.) In Geometry we still differentiate between **constructing**, **drawing**, and **sketching**. In a drawing, **rulers** and **protractors** are allowed, whereas a sketch may be a free-hand representation.

The Greeks quickly mastered many constructions, such as for the regular pentagon, perpendicular bisector, equilateral triangle, *etc.,* which must still be learned by high school geometry students. However, try as they might, they came up with four which defied solution. These **four unsolved problems of antiquity** remained so until the 1800's. They are:

1. Squaring a circle (construct a square with area equal to a given circle);
2. Duplicating a cube (construct a cube with twice the volume of a given cube);
3. Trisecting an arbitrary angle;

4. Constructing a regular heptagon (or actually all regular polygons).

During the 1800's, advances in mathematics enabled mathematicians to prove them all unsolvable under the construction rules then in vogue. An important part of the solution was to couch the problem in terms of algebraic, rather than geometric terms. One soon discovers that constructions with straight-edge and compass represent rational operations and square roots, but not cube or higher roots. Thus if a cube root is unavoidable, the construction is impossible. The algebraic equations involved have what are known as **algebraic** roots.

In 1844 the French mathematician Joseph Liouville (1809–1882) proved **nonalgebraic** or transcendental numbers existed. His proof was not simple, but allowed him to produce several examples, the most famous is known as **Liouville's number** and can be written either as $0.110001000000000000000001\cdots$ or $10^{-(1!)} + 10^{-(2!)} + 10^{-(3!)} + 10^{-(4!)} + \cdots$. Another favorite example is $0.1234567891011\cdots$, where the natural numbers occur in order. Integers of this form are known as Smarandanche Concatenated Numbers and work on their prime factorization can be viewed here.[2]

Although it had been already shown in 1737 by Euler that e and e^2 and in 1768 by Lambert that π were all irrational, it took many more years before they were proved to be transcendental.

In 1873, Charles Hermite (1822–1901) proved e was transcendental.

He wrote "I shall risk nothing on an attempt to prove the transcendance of π . If others undertake this enterprise, no one will be happier than I in their success. But believe me, it will not fail to cost them some effort."

But in 1882, Ferdinand Lindemann (1852–1939) proved π was transcendental and coined the term.

Transcendental numbers are irrational numbers that are not the roots of algebraic equations.

The transcendance of π finally solved, all-be-it in the negative, the problem of squaring the circle. Since π is not algebraic, a segment of length the square root of π is impossible to construct.

In 1795 Gauss proved that it is possible to divide the circumference of a circle into n equal parts when n is odd, if n is either a prime Fermat number or a product of different prime Fermat numbers. He was 18. It was published in 1801 in his major work *Disquisitiones aritmeticae*.

In 1837 Wantzel published a proof that no other regular polygons can be constructed, thus settling in the negative the question of the constructability of the regular heptagon. However, the regular heptadecagon (17-gon) is constructable! Wantzel

[2] `http://www.worldofnumbers.com/factorlist.htm`

also proved that the angle of 60° was not trisectable since the equation $4x^3 - 3x = 1/2$ has no roots which are rational or rational combinations of square roots. Wantzel is also responsible for the developments proving that the cube root of 2 is also not constructable with the same year usually given.

15.6 Many More Transcendentals

Although π and e are the two most famous transcendental numbers, there are plenty more. Just as the reals can be divided into two disjoints sets, *i.e.* the rationals and irrationals, the irrationals (or reals) can be similarly subdivided into algebraics and transcendentals. Another way to classify the real numbers is as any number that can be written as a decimal fraction. These decimals are of three types: 1) terminating; 2) nonterminating but repeating; and 3) nonterminating, nonrepeating. We explored the terminating and repeating decimals in Numbers Lesson 9 and concluded they were all rational numbers. This last class, however, is another way to characterize the irrational numbers.

There are more irrational numbers than rational numbers.

This is fairly clear since the rational numbers were denumerable, but the real numbers, made up of the rational numbers and irrational numbers, were nondenumerable.

Logarithms and the trigonmetric functions are examples of **transcendental functions** introduced and studied in the high school math curriculum.

Algebraic numbers are enumerable! Almost all real numbers are transcendental.

It has been very difficult to prove numbers to be transcendental. David Hilbert (1862–1943) challenged the mathematical community in 1900 with a list of **23 unsolved problems** in mathematics of utmost importance. In fact, the quote used to open this lesson came from this speech. The seventh problem was to prove that for any algebraic number ($a \neq 0$ or 1), and any irrational, but algebraic number b, a^b is always transcendental. The first in 1929 and the second a year later, the Russian mathematician Gelfond proved Hilbert's two examples, $e^\pi = i^{-2i}$, and $2^{\sqrt{2}}$ to be transcendental and in 1934 proved the general case.

The status of many numbers remains unknown: π^π, e^e. Others: π^e, 2^e, and 2^π have not even been proved to be irrational! The $\sin 1°$ is algebraic, whereas $\sin(360°/2\pi) = \sin(1\,\text{rad}) = \frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} - \frac{1}{11!} \cdots$ is transcendental.

15.7 Transcendental Homework

Each problem is worth two points, except as noted.

1. Evaluate the following rational number and compare it relative to e: $\dfrac{58,291}{21,444}$.

2. Evaluate the following rational number and compare it relative to e^2: $\dfrac{158,452}{21,444}$.

3. Find a decimal approximation for the real number halfway between e and π.

4. Find a decimal approximation for the real number halfway between π^e and e^π.

5. Find the circumference of a circle with diameter of 7", using the approximation $\pi \approx 22/7$.

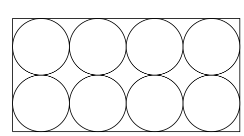
6. Find the **exact** and **approximate** area for a circle with radius 5 m. (Be sure to include proper units!)

7. Give, to the nearest hundredth square foot, the area that can be irrigated by a circular sprinkler that spouts water 60' as it rotates around a fixed point. Give the circumference of the region to the nearest tenth foot.

8. A circle has area 100π in^2. Find the **exact** radius, diameter, and circumference.

9. On a 12" pizza, what does the 12" refer to? How many times as much of each ingredient is needed for a 16" pizza with the same thickness? What is the area of each slice when a 16" pizza is divided evenly among 6 people? (see textbook 8.9:13).

10. Eight metal disks equally, but maximally sized, are cut out of a metal sheet 18" by 36". The rest is not used. What is the area of the metal that is not used? What percent of the metal is used? (see textbook 8.9:14).



11. Find a can or bottle with a circular base. Measure the diameter (d) as accurately as possible. Measure the circumference (C) with a tape measure or by rolling the can on the ruler. Calculate the C/d ratio to the nearest hundredth. What number should it approximate? Explain any difference?

12. A sheik dies with 3 sons and 17 camels. Earlier he had told his steward to give the youngest son 1/2 his camels; his middle son 1/3 his camels; and his oldest son 1/9 his camels. Without any fractional camels, how did the steward do it? How many camels did each son get? (This is a puzzle question.)

13. Find which ordinal number corresponds to Andrew Jackson's presidency (as in which president was he?) and what year he was first elected. Relate this information to the number e.

14. Add the first, then second, then third, ... **terms** in the following **sequence**: $\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}+\cdots$ What transcendental number does the resulting **series** appear to approach?

15. Add the first, then second, then third terms, $\cdots$ in the following sequence: $\frac{4}{1}-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}+\frac{4}{9}-\frac{4}{11}+\frac{4}{13}-\cdots$. What transcendental number does the resulting series appear to approach (from above and below! and allbeit very slowly)?

16. (**Three points:**) Convert $57°$ and $196°$ into radians and $5\pi/9$ into degrees.

17. (**Three points:**) Evaluate: $\log_3 81, \log_{10} 100$, and $\log_9 3$ without using a calculator.

18. Convert $\log_4 x = 3$ into exponential form and solve for x.

19. Read sections 8.8 and 8.9 in your geometry text. See problems 8.8: 3, 5, 12; 8.9: 1, 5, 11, and 12.

20. **Bonus:** Look up **Napier's Bones** or **Napier's Rods** in an encyclopedia or dictionary. What were they? How many were there? What did they look like? How did they work? What specie bone were they?

Numbers Lesson 16

Imagine More Complex Numbers

$e^{i\pi} + 1 = 0$ is the most remarkable formula in mathematics Feynman

This lesson motivates the complex numbers as solutions to certain polynomials and introduces them as the cartesian product of the reals and imaginaries. Complex numbers are added, subtracted, multiplied, divided, and their magnitude found. We graph them, introduce the polar form, and find roots in that form. We close with a list of the Greek alphabet and a summary of errata and future improvements.

16.1 Father of Complex Powers: Abraham de Moivre

Abraham de Moivre (1667–1754) was born in France but moved to England while a teenager for political refuge (after the law protecting protestants was lifted). There he chanced to met Newton's *Principia Mathematica* and supported himself by lecturing and tutoring. He soon established himself as a respected first-rate mathematician and was elected to the Royal Society in 1697. He was eventually asked to decide between Newton and Leibnitz regarding the invention of the calculus, in a process some say was rigged. de Moivre never obtained a permanent teaching position, although his research on probability was sought after as a consultant for both life insurance and gambling. He outlived his friends, dying the relative poverty which plagued his life. His name lives on in **de Moivre's Theorem** given later in this lesson.

16.2 The Complex Numbers

It would seem that with so many real numbers, mathematicians would be satisfied. However, just as negative numbers allowed us to solve equations such as $x+a=0$, so too do imaginary numbers, or more accurately complex numbers, allow us solutions to all quadratic and higher degree polynomial equations. The choice of the term imaginary has been somewhat unfortunate, but with exposure and practice, these numbers can become just as meaningful as the reals. Consider the following solution.

$$\begin{aligned} x^2+1 &= 0 \\ x^2 &= -1 \\ x &= \pm\sqrt{-1} = \pm i \end{aligned}$$

$i = \sqrt{-1}$ is termed the **unit imaginary**—all imaginary numbers can be formed as multiples thereof.

For most students, the first exposure to complex numbers is in solving quadratic equations that have no real solutions, such as $x^2 - 4x + 5 = 0$. Using the quadratic formula, we find that the discriminate (the part of the formula under the radical) is negative (-4)—but how do we take the square root of -4? Using this new symbol $i = \sqrt{-1}$, and our rules for manipulating radicals, it becomes $x = \sqrt{4}\,i = 2i$, and the solutions to this equation are the complex numbers: $2 \pm i$. The rules for adding and multiplying complex numbers are given below, but if your calculator is in $a+bi$ mode, you can check this result on it by typing: $(2+i)^2 + (2+i) + 5$ or $(2-i)^2 + (2-i) + 5$ and obtaining the result of zero.

Complex numbers are of the form $a + bi$, where $a \in \mathcal{R}$ and $b \in \mathcal{R}$.
a is called the **real part**, and b (**not** bi) is called the **imaginary part**.

Real and imaginary numbers are both "small" subsets of the complex numbers. Real numbers are represented by a, where $b = 0$. Whereas, when $a = 0$, $a + bi$ is just bi—the imaginary numbers. The complex numbers are represented by the symbol $\mathcal{C}$. A common mistake is to refer to the complex numbers as the imaginary numbers. However, the imaginary numbers are only a very special subset of the complex numbers. The term **non-real complex** is often used, since all real numbers are complex numbers.

Cantor showed the unbelieveable fact that points in a unit square could be mapped to the points in a unit line segment, as noted earlier in his biography (9.1). This procedure can be used to put the complex numbers into a one-to-one relationship with the real numbers, thus showing their size to be the same non-denumerable infinity!

$$\mathcal{N} \subset \mathcal{Z} \subset \mathcal{Q} \subset \mathcal{R} \subset \mathcal{C}$$

The **complex conjugate** of $a + bi$ is $a - bi$.

Complex numbers often appear in **conjugate pairs**—see the quadratic formula for why. i can be treated just like a variable, such as simplifying powers:

$$\begin{aligned} i^0 &= 1 \\ i^1 &= i \end{aligned}$$

$$\begin{aligned}
i^2 &= -1 \\
i^3 &= i^2 i = -1 \cdot i = -i \\
i^4 &= (i^2)^2 = (-1)^2 = 1 \\
i^n &= i^{n \bmod 4}
\end{aligned}$$

16.3 Operations with Complex Numbers

Your TI-84+ graphing calculator will do extensive calculation with complex numbers. (Check your **MODE** and be sure you are in $a + bi$ and not **Real** or $re^{i\theta}$.)

16.3.1 Adding or Subtracting Complex Numbers

Add or subtracting complex numbers involves adding/subtracting like terms. (Don't forget subtracting a negative is adding!)

$$\begin{aligned}
(3-2i)+(1+3i) &= (3+1)+(-2i+3i) = 4+1i = 4+i \\
(4+5i)-(2-4i) &= (4-2)+(5i+4i) = 2+9i
\end{aligned}$$

16.3.2 Multiplying Complex Numbers

To multiply complex numbers treat them like binomials and use the FOIL method, but simplify i^2.

$$\begin{aligned}
(3+2i)(2-i) &= (3 \cdot 2)+(3 \cdot -i)+(2i \cdot 2)+(2i \cdot -i) \\
&= 6-3i+4i-2i^2 \\
&= 6+i-2(-1) \\
&= 8+i \\
(2+i)^2 &= (2+i)(2+i) = 4+4i-1 = 3+4i
\end{aligned}$$

$\sqrt{-9} \cdot \sqrt{-16} = i\sqrt{9} \cdot i\sqrt{16} = i^2 \cdot 3 \cdot 4 = -12$. Notice how our order of operation is important (exponentiation before multiplication) as commonly the incorrect answer $\sqrt{144} = 12$ is obtained.

$$\boxed{\text{If } x > 0, \text{ then } \sqrt{-x} = i\sqrt{x}.}$$

16.3.3 Dividing Complex Numbers

To divide complex numbers, multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{2+3i}{3+i} = \frac{(2+3i)(3-i)}{(3+i)(3-i)} = \frac{6-2i+9i-3i^2}{9-i^2} = \frac{6+7i+3}{9+1} = \frac{9+7i}{10} = 0.9+0.7i.$$

16.3.4 Magnitude

To find the magnitude of a complex number you find its distance to the origin: $|3+4i| = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$.

Magnitude is often confusingly referred to as absolute value, since the same symbol is used. In fact, you must use `abs` on your TI-84 calculator! Notice how both are a measure of distance and the Pythagorean Theorem is used here. A common mistake is to include the i under the radical—avoid that error.

16.4 Graphing Complex Numbers

Complex numbers are graphed on the **complex plane**—the cartesian product of the reals and the imaginaries. As such, it is very similar to the xy-plane. The familiar x-axis is still the familiar real number line and the y-axis is replaced with a number line containing the imaginary numbers. This is often termed an **argand diagram.** Cantor showed it was possible to construct a one-to-one correspondence between every point in the plane and the real number line. On a unit square one can map the ordered pair with decimal expansion $(0.a_1a_2a_3\cdots, 0.b_1b_2b_3\cdots)$ to the real number $0.a_1b_1a_2b_2a_3b_3\cdots$ thus interleaving the decimal expansions. Thus, it would seem, the complex numbers have the same cardinality as the reals.

16.5 Polar Form

Complex numbers are also often located on the complex plane by their distance from the origin and angle from the positive x-axis. The angle might be given in either degrees or radians. What your TI-84+ calculator uses is controlled both on input and output by mode. However, unlike the trig functions, putting the degree symbol on an angle does not override radian input! By setting $a+bi$ or $re^{i\theta}$ (polar) format and inputting the alternate form, it will interconvert for you.

The following relationship named after Euler is often used:

$$Ke^{i\theta} = K(\cos\theta + i\sin\theta),$$

where sin and cos are the trigonometric relationships discussed in Numbers Lesson 12. Thus if $K=1$ and $\theta = \pi/2 = 90°$, the complex number located one unit directly above the origin is obtained. This is i, because $\sin 90° = 1$ and $\cos 90° = 0$. r is a much more common choice of variable to represent magnitude, but the author feels the choice of K will be much more meaningful and memorable for his students!

16.6 Greek Alphabet

The table of Greek letters below with names and phonetic English equivalents should be committed to memory by the grade A math and science student.

lower	**upper**	**name**	**equivalent**	**lower**	**upper**	**name**	**equivalent**
α	A	alpha	a	ν	N	nu	n
β	B	beta	b	ξ	Ξ	xi	x
γ	Γ	gamma	g, n	o	O	omicron	o
δ	Δ	delta	d	π	Π	pi	p
ϵ	E	epsilon	e	ρ	P	rho	r, rh
ζ	Z	zeta	z	σ	Σ	sigma	s
η	H	eta	$\overline{e}$	τ	T	tau	t
θ	Θ	theta	th	υ	Υ	upsilon	y, u
ι	I	iota	i	ϕ	Φ	phi	ph
κ	K	kappa	k	χ	X	chi	ch
λ	Λ	lambda	l	ψ	Ψ	psi	ps
μ	M	mu	m	ω	Ω	omega	$\overline{o}$

16.7 Finding n^{th} Roots

de Moivre's Theorem states that $z^n = k^n \operatorname{cis}(n\theta)$, where

$\operatorname{cis}\theta$ is an abbreviation for $\cos\theta + i\sin\theta$.

n may be fractional thus $z^{1/n} = k^{1/n}\operatorname{cis}([\theta + 360j]/n)^\circ$, where j is an integer ranging from 0 to $n-1$. We can apply this to the multiplicative identity (1) which also has a magnitude of 1. It is clear 1 has two square roots: ± 1. Since -1 has two square roots, it should now be clear that 1 has four fourth roots: ± 1 and $\pm i$. We can apply de Moivre's Theorem to obtain the eight eighth roots as follows.

The **Eight Eighth Roots of Unity** are ± 1, $\pm i$, $\pm\sqrt{2}/2 \pm i \cdot \sqrt{2}/2$. (This last expression is generally considered ambiguous as to how many points it represents, but here represents four distinct points.) Note how they are very symmetrically arranged (on a circle) on the complex plane. Note also how the radical relates to $\sin(45^\circ + 90^\circ n)$ and $\cos(45^\circ + 90^\circ n)$.

16.8 Errata

Students should organize their booklets for stapling now. Check to be sure you have all your pages in page number order. An occasional funny page sequence will occur. Lessons 12 and 15 had a odd number of pages and a page will be "missing" (ix, x, 104, and 134). These were not replaced with something else this year. Various pages in the appendices (title, activities, quizzes, keys) have been omitted this year.

This will force homework to be interleaved! You might have additional homework pages and it is your choice where these are **neatly** located. **Be sure** you have the box (A.5), complex number (A.6), and booklet (A.7) activities, and 1 quiz (B.1). Do **not** have your test or test key stapled within the booklet (but the released tests (C.1 and ?[1]) ARE part of the booklet).

Several problems were fixed and figures added in this revision—many after printing, however. A summary of recent/future changes follows.

- Consider distributing the lessons as school starts next year.
- Many converted activities (set, dice, factors, magic boxes) remain difficult to squeeze in. Some remain unconverted (TI-84 intro, calculator fractions, fraction matchup, calculator slopes, 24, logs) but may have been moved into the lesson or into summer algebra.
- Lesson 12 could be split between Pythagoras and Fermat and the bios expanded for Diophantus and Goldbach. Galileo's bio was moved to Stats, perhaps temporarily—I need his quote! The first part of 13 could go with the new lesson.
- The early lessons were split up in 2008 to add a lesson but at least one homework question was moved after printing in 2009. We have not yet moved the other 4 Peano Axioms here. The well-ordered axiom/axiom of choice is mentioned in both lesson 3 and 14. Euclid's algorithm could be added. Maybe some odd questions can be repeated as evens in later lessons.
- Lessons 6 and 7 remain at 6 pages but tend to be dense. Breaking this streach up could help things as well. Pascal's bio needs a better place near here.
- Odd solutions should be generated from the beamer/pdf work and made available. The software calculator (TI-SmartView) was used very little.

16.9 Epilogue

This document is not yet a finished product—improvement and corrections are an ongoing process. With this fourth pdf version the old html version has been removed from the web, except for the odd solutions. It is, however, a dream come true. Some work remains to smooth out areas like logic and paradoxes, even out the level of effort required, and make the homework do what I want it to. It is planned for Center students to take some responsibility to clarify the less clear and extend the more interesting aspects. Continued feedback is appreciated.

[1] Not yet labelled and integrated.

16.10 Complex Homework

Perform the following operations with complex numbers: (Show work!! Only use a calculator to check your answer.) **Each problem is worth two points, except for problems 6 and 12 which are 5 points each.**

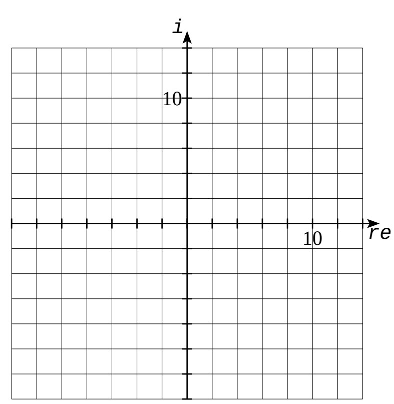
1. $(3 + 5i) + (8 + 9i) =$

2. $(4.5 + 3i) + (3 - 1.5i) =$

3. $(7 + 13i) - (8 + 2i) =$

4. $(-5 + 3i) - (3 - 8i) =$

5. $(-3i) - (13 + 4i) =$



6. Graph the answers to the problems 1–5 on the grid above.

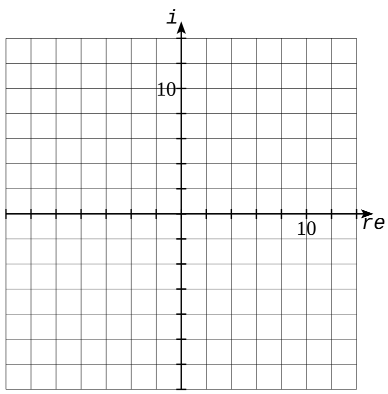
7. $(1+2i)(1-2i) =$

8. $(2-3i)(-3+2i) =$

9. $(3+2i)^2 =$

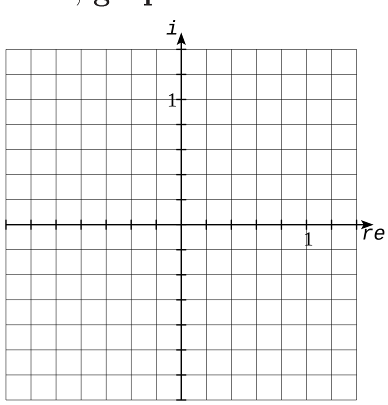
10. $(6+8i) \div (1+3i) =$

11. $|(3+5i)| =$



12. Graph the answers to the problems 7–11 on the grid above.

13. Assuming the **cube roots of 1** are equally space around the unit circle, you know the real one (1), and the two complex ones are complex conjugates of each other; **graph them and find approximate values** for them.



14. Refine your values for the problem above using the exact trigonometric values in the table on page 12.4 in Numbers Lesson 12 and check them on your calculator.

Appendix A

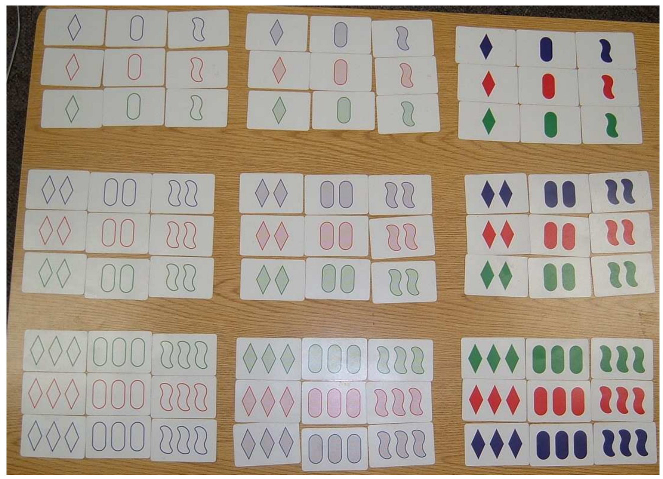
Activities

A.1 Activity: Set Game and Crossword Puzzle

On the back side is a crossword puzzle using the vocabulary words below.

The game of Set is a useful way to explore the meaning of this undefined word in mathematics.

A set deck consists of 81 cards—all different. There are 81 cards because on four different properties: color, number, shape, fill, they have three different states. The colors are: red, green, and purple. The number of identical shapes on a card is either one, two, or three. The shapes are: diamond, oval, and squiggle. The fill patterns are: filled, hashed, and empty. sometimes referred to as solid, liquid, and gas.



The object of the game is to find three cards which for each of these four characteristics (properties) are either all the same or all different. A good rule to use is: if there are exactly two of something, it isn't a set.

Let's play a little set (available online. In the game of SET, you will form sets of 3 cards as described above.

One person at each table will act as the dealer and deal 15 shuffled cards face up on the table. Players will initially take turns and after selecting 3 cards, explicitly tell whether each of the 4 aspects are the same or different. Magic rule: if 2 are the same, but the third is different, it is not a set. After the card stack is depleted, players will **display** their sets and especially **call attention** to any set with 3 or 4 different aspects.

Tally points for each set: 1 point for each different characteristic. For example: If you have three diamonds on each card with each a different color and shade, the set will be two points. The person with the most points wins. (If all the groups are competing, the table with no cards unsetted will get an extra five points for their members.)

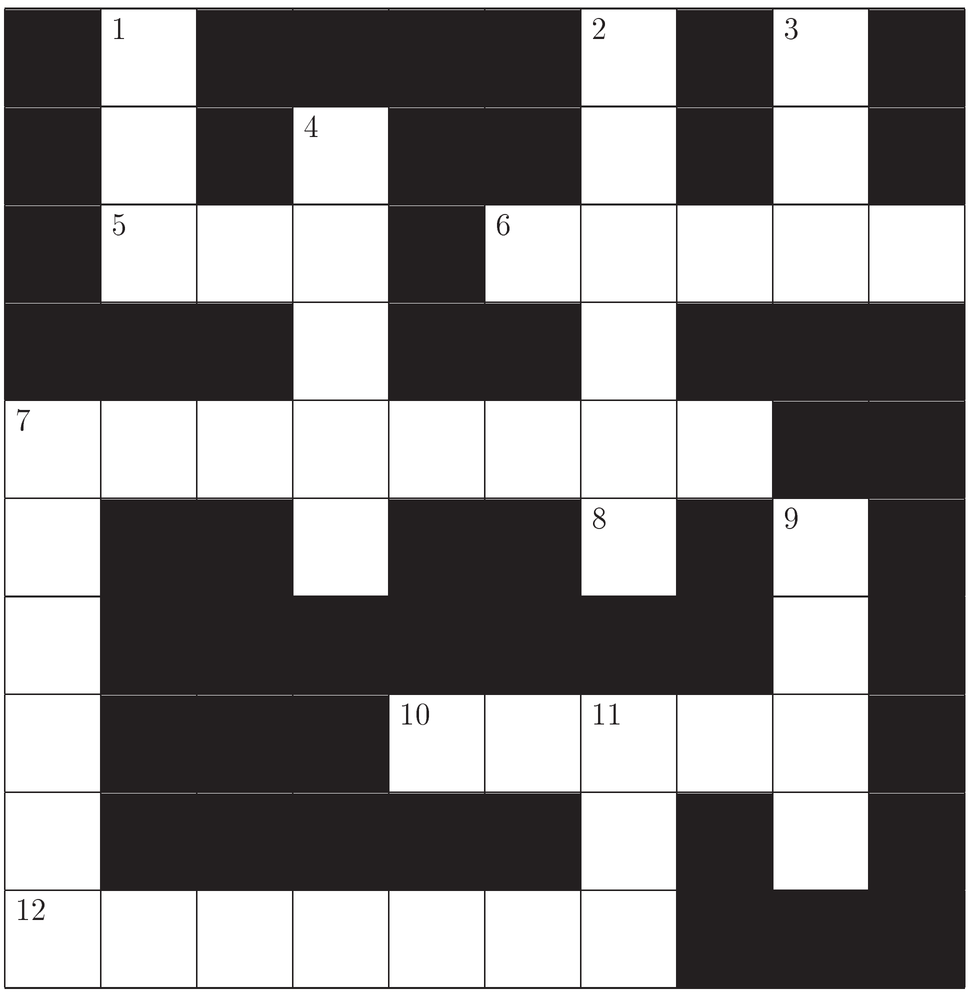
Name ______________________________ Score ________

Across

5. Homophone of to and too.
6. Color of grass, money, *etc.*
7. Rhymes with jiggle and not quite **oval**.
10. More than 2 and less than 4.
12. Women's best friend.$^\diamond$

Down

1. Ready, _____, Go.
2. Red and blue make _________.
3. _____ is the lonliest number . . . .
4. State of matter (not gas nor liquid).
7. What the tree did to the house all summer.
9. If the door is not closed . . . .
11. _________ for the blood we shed.



A.2 Counting Activity: Skittles

- Divide a 16 ounce (one pound) package of Skittle^{TM}; brand candies approximately equally into 7 paper cups.
- Assign each cup to a group. Each group must tally each color and record their data on the chart below. PLEASE do not destroy any evidence until you have double checked your results. Do not contaminate the specimens.

	Yellow	Orange	Red	Green	Purple	Total
Table 1						
Table 2						
Table 3						
Table 4						
Table 5						
Table 6						
Table 7						
Total						

In 2002 there was no yellow, but white, a mystery flavor.

- Discuss variations of the data.
- Be sure to turn this sheet in at the end of the class period.

We will assemble this data and you will use it again in a few weeks for statistics.

A.3 Factor Activity

```
Open books and open table quiz.  Hand in one per table.
Accepted only when the answers are correct.  Keep a copy in your notes.
```

1. Find all the factors of 18.

2. Add all the factors of 18, except for itself.

3. Find all the factors of 30.

4. Add all the factors of 30, except for itself.

5. Find all the factors of 42.

6. Add all the factors of 42, except for itself.

7. Find all the factors of 54.

8. Add all the factors of 54, except for itself.

9. What is the pattern?

10. Does it continue?

A.4 Magic Boxes (Base 2) Activity

1	3	5	7
9	11	13	15
17	19	21	23
25	27	29	31

2	3	6	7
10	11	14	15
18	19	22	23
26	27	30	31

4	5	6	7
12	13	14	15
20	21	22	23
28	29	30	31

8	9	10	11
12	13	14	15
24	25	26	27
28	29	30	31

16	17	18	19
20	21	22	23
24	25	26	27
28	29	30	31

Each table must select <u>one</u> person to be their facilitator. This designation may persist for several weeks until new seating or other needs determine a change. The facilitators leave the room and are given this instruction sheet. An assistant will go over with them the instructions on the back, and have them return to their table.

Each facilitator must ask table members in turn to secretly pick any number between 1 and 31. Table members point to each box beside in which their number appears. The facilitator will then tell them their secret number!

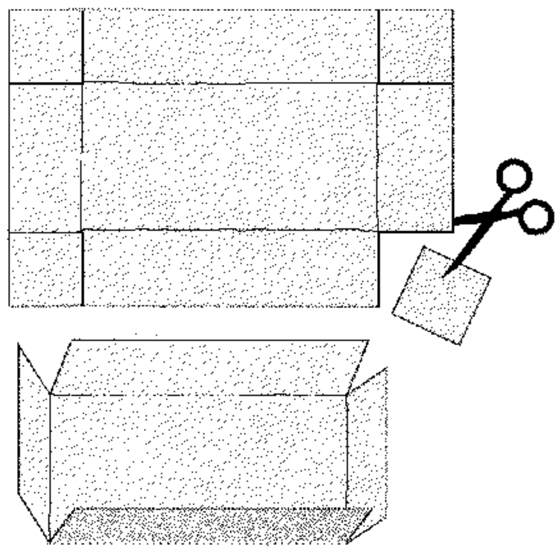
After each person gets at least one turn, the facilitator will try to help table members understand how the trick works.

Solution: Add up the first number in each of the boxes the person chose.
The number you calculated is the same as they have chosen.

Reasoning: The first number in each box is a power of two. 1, 2, 4, 8, and 16. Each box represents the power: box 0 is 2^0, box 1 is 2^1, box 2 is 2^2, *etc.* The numbers have been arranged in each box such that the combination of the powers will correspond to its binary representation. For example 19 is equal to 16 + 2 + 1, or $19 = 10011_2$, and you'll find 19 in what we will call box 0 (2^0=1), box 1 ($2^1 = 2$), and box 4 ($2^4 = 16$).

A.5 Maximal Box Volume Activity

Task: Given a piece of paper 8" × 10", find all dimension**S** of the box (no top) with the largest volume which can be formed by removing equal squares from each corner and folding up the resulting tabs on each side.



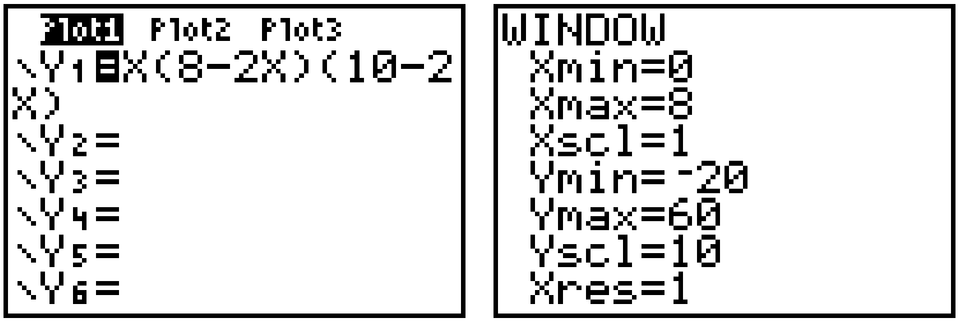
A.5.1 Method I (scissors and water)

Use scissors and trial and error. (Sorry, no water will be provided.)

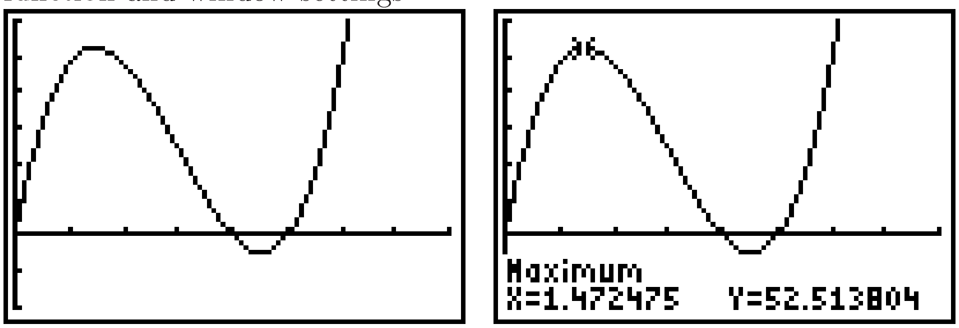
A.5.2 Method II (TI-84 graphing calculator)

Volume = height × width × length
$V = x \times (8 - 2x) \times (10 - 2x)$
Press the **Y=** key and enter the equation (with Y_1 being V above).

Press the **WINDOW** key and enter the following:
$Ymin = 0; Xmax = 8; Xscl = 1; Ymin = -20; Ymax = 60, Yscl = 10$
Press the **GRAPH** key.
To find the maximum value in the graph pressing **CALC** key (2$^{\text{nd}}$ TRACE). Press **4** for **maximum**.
Once you request maximum, "LeftBound?" appears on the screen. Arrow over to the left side of the maximum. Press **ENTER**. "RightBound?" now appears. Arrow over to the right side of the maximum and press **ENTER**. "Guess?" now appears. Arrow toward the maximum and press **ENTER**.
The screen shows the maximum volume possible ($y =$) and the corresponding x value. Finish by calculating the other dimensions. What is the meaning of the negative volume?



function and window settings



graph of function and maximum

A.5.3 Method III (very simple calculus)

simplify: $V = x(80 - 36x + 4x^2)$
$V = 80x - 36x^2 + 4x^3$
$V' = 80 - 72x + 12x^2$ (To find the slope of a polynomial at any point, multiply the exponent by the coefficient and put it down as the new coefficient. Write down your variable with the exponent reduced by one. If there is no variable, the slope is zero, so don't write anything for that term.)
$V' = 3x^2 - 18x + 20 = 0$ (rearranged order, $\div 4$, and set V' to zero because slope is zero at a maximum.)

$x = (18 \pm \sqrt{324 - 240}) \div 6$ (Use the quadratic formula to solve the resulting quadratic equation.)
$x = 3 \pm \sqrt{21}/3 \approx 1.47247 \cdots$

Thus the other sides are $(8 - 2x) \approx 5.055$ and $(10 - 2x) \approx 7.055$.

Note: factorable quadratics and integer solutions can be obtained by starting with square paper.

Note also: this is the solution to the third bonus questions (either question number 43 or 83) of the May 1998 semester tests (Geometry, Algebra II, Precalculus). It also appeared on that year's Calculus AB final test.

A.6 Complex Number Activity (Lesson 16)

Please use your TI-84+ calculator or TI-$n$spire with TI-84+ keypad for the following activities. How to do many of them without your calculator is illustrated in the lecture notes.

Find the i key on your calculator (**2nd**) and (**.**) and **ENTER** The answer should be i or possibly $1e^{90i}$.

$\sqrt{-1}$ and **ENTER**

Don't be surprised with an error.

MODE set $a + bi$ and **ENTRY** (2nd ENTER) and **ENTER**.

The answer should now be i. **Real** mode may be safest until you understand what it is trying to do!

MODE $re^{\theta i}$ $\sqrt{-1}$ and **ENTER** Your answer should be $1e^{90i}$.

Set your **MODE** back to $a + bi$.

(3 - 2i) + (1 + 3i) **ENTER** should give you: $4 + i$.

$(3 + 2i) * (2 - i)$ **ENTER** should give you: $8 + i$.

Note: the multiplication sign is optional.

$(2 + 3i)/(3 + i)$ **ENTER** should give you: $.9 + .7i$.

abs$(3 + 4i)$ (**MATH NUM 1**) should give you: 5.

Note: the calculator uses abs for both absolute value and magnitude.

i^i **ENTER** should give you .2078775764!

Amazing! Imaginary to imaginary give you a real number. Actually, this is only the primary answer, other values are also possible.

i^{-2i} **ENTER** and e^{π} **ENTER** both should give you 23.14069263.

sin(i) and cos(i) should give you an error on the TI-83 and TI-84, but works properly on the TI-85 and TI-86.

A.7 Numbers Booklet Verification/Stapling Activity

Directions: You may work together, but answer each question carefully using **your own** Numbers booklet. Take time to put the booklet in **THIS** order. Make a list by **table** of who is missing what (nonbonus) pages.

1. Page *i* (blue front cover): Full title of booklet.
2. Page *iii*: Title for Section 2.5.
3. Page *v*: Section number for Accuracy *vs.* Precision.
4. Page *vii*: Title for Section A.4.
5. Page *xii*: "Convey my lifelong ___ ___ ___ ___ ___ ___ ___ for numbers."
6. Page 2: Q7: Leave textbook home until when?
7. Page 7: John Venn's year of death.
8. Page 9: Q5. Cost of new toy in clams.
9. Page 12: Another word for axiom (top of page).
10. Page 19: Eratosthenes' nickname (bottom of page).
11. Page 25: Who said "Ah! I recognize the lion by his paw."
12. Page 31: Q5. Largest factorial calculated exactly on your TI-84 calculator.
13. Page 35: Restriction on "Anything to the zero power is 1."
14. Page 41: Q2. Zeroes in a googolplex.
15. Page 43: Latin quote from Decartes.
16. Page 49: Q8. Counterexample to large dangerous bears.
17. Page 56: What `I.OU6.(O4.O5.NO6)` is equal to (middle of page)?
18. Page 58: Q9. Objects headed toward St. Ives (in base 7).
19. Page 60: Group axiom 1.
20. Page 66: Q20. $2^{2^5} - 1$ in hexadecimal.
21. Page 70: Done when multiplying/dividing inequality by a negative (middle of page).

22. Page 76: Q12.13 Repeat length for $\frac{1}{13}$.
23. Page 83: Why isn't 12 am or 12 pm valid (middle of page).
24. Page 85: Q14. $2.4526\,\text{m} \div 8.4$.
25. Page 89: What Q.E.D. means (middle of page).
26. Page 93: Q6 (go all the way!).
27. Page 97: What is special about a $37°, 53°, 90°$ triangle (middle of page)?
28. Page 102: Q6. Length of other two sides in $30°, 60°, 90°$ triangle.
29. Page 106: Number system used to label quadrants (bottom of page).
30. Page 114: Q16. Domain and range of: $y = x^2 + 5x + 6$.
31. Page 120: What is the continuum hypothesis?
32. Page 123: Q8. Counterexample showing negatives are not closed under multiplication.
33. Page 127/129: What decimal place has the first identical digit in the decimal representations of π and e?
34. Page 133: Q18. Solve for x: $\log_4 x = 3$.
35. Page 139: Three pages "missing" page numbers (bottom of page).
36. Page 142: Q14. Exact/approximate values of complex cube roots of 1.
37. A.2 Bonus: Page 146: Total skittles for your (original) table group.
38. A.5 Page 150: Square side length to cut in 8" $\times$ 10" corners to maximize volume (page 2 screen).
39. A.4 Bonus: Page 148: Which boxes have 31 in them (specify by number in upper left)?
40. A.7 Bonus: Page 153: Express 1/(last question number) exactly as a decimal fraction.
41. B.1 Page 158: Q9. 100 expressed as sum of two triangular number.
42. C.1 Released test: Page 161: Q8 LCM(270, 600).
43. C.2 Released test: (page 4): Q17. 4 ancient impossibilities.

Your booklet should now be ready for stapling. Bonus for early.

A.8 Number/Phrase Association Activity

`Complete the phrase identified by these numbers, words, and initial letters.`

1 - D at a T	1 - W on a U	
2 - T D (and a P in a P T)		
3 - P for a F G in F	3 - B M (S H T R)	3 - L K
4 - H of the A	4 - Q in a G	4 - T on a C U
5 - D in a Z C	5 - F on the H	6 - W of H the E
7 - H of R	7 - W of the A W	7 - V of S
7 - D (with S W)	7 - B M and the E	
7 - S	7 - D S	
8 - P on N A	8 - P of S in the E L	8 - S on a S S
9 - I in a B G	9 - P in the S S	9 - J of the S C
10 - A in the B of R	10 - C in the D	
11 - P on a F T	12 - S of the Z	12 - D of C
13 - C in a S	13 - S on the A F	13 - D in a B D
16 - O in a P	16 - M on a D M C (YHH and a B of R)	
18 - H on a G C	18 - W on my B R	
20 - Y that R V W S	24 - H in a D	
26 - L of the A		
29 - D in F in a L Y		
30 - D H S A J and N		31 - I C F at B-R
32 - D F at which W F		
36 - I in a Y		
40 - T (with A B)	40 - D and N of the G F	
50 - C in a H D	50 - W to L Y L	
54 - C in a D (with the J)		
56 - S of the D of I		57 - H V
60 - S in a M	64 - S on a C	
66 - B in the B	76 - T in the B P	
80 - D around the W	88 - K on a P	90 - D in a R A
99 - B of B on the W		
101 - D	200 - D for P G in M	
212 - D at which W B		
435 - M of the H of R		
500 - H of B C	600 - R in the C of the L B	
1000 - W that a P W	1000 - S (that a F L)	
1001 - A N		
20,000 - L U the S		

Appendix B

Quizzes

Name ______________________________ Score ________

B.1 Quiz over Numbers Lessons 1–4

`Open books and open group` **quiz.**	`Hand in one per table.`
`Be sure answers are correct!`	`Keep a copy in your notes.`

1. List table members who **do not** have their syllabus signed by a parent.

2. Set intersection and union are related to **and**'s and **or**'s. Which is which and why?

3. List one quote by each of the three greatest mathematicians and indicate whose is which.

4. What is your group's best answer for Numbers Lesson 1, problem 9?

5. What is your group's best answer for Numbers Lesson 1, problem 10?

6. Show work for Numbers Lesson 2, problem 7.

7. What is your group's best answer for Numbers Lesson 3, problem 8a?

8. What is your group's best answer for Numbers Lesson 4, problem 7?

9. Express the number 100 as the sum of two triangular numbers.

10. List five common Latin terms and what they mean.

Appendix C

Released Test/Key

Name ______________________________ Score ________

C.1 Geometry, Test 1, September 24, 2004—Released Test

```
One 3"x5" notecards and TI-84+ type graphic calculator allowed.
Please place answers on the short underlines provided to the
left of the problem symbol.  Each of the 21 question numbers
has equal weight (i.e. 5 points each).  Question subparts have
about equal weight.  Read the questions carefully.  Hand in any
used scratch paper with the test for potential partial credit.
SHOW YOUR WORK
```

5

1. Form the best **match** among the following.

_____	Triangular Numbers	A.	$0, 1, 4, 9, \ldots$
_____	Squares	B.	$0, 1, 1, 2, \ldots$
_____	Perfects	C.	$0, 1, 3, 6, \ldots$
_____	The Factorials	D.	$6, 28, 496, \ldots$
_____	The Fibonacci Numbers	E.	$1, 1, 2, 6, \ldots$

5

___ **2**. Perform the following set operation: $\{B, r, i, t, n, e, y\} \cap \{S, p, e, a, r, s\}$. (**Three bonus points:** what is the cardinality of each set?)

5

___ **3**. Perform the following set operation and sketch the corresponding Venn diagram. $\{B, r, i, t, n, e, y\} \cup \{S, p, e, a, r, s\}$.

5

___ **4**. Explicitly use the recursive definition of $n!$ to simplify then evaluate: $\frac{7!}{4!}$.

5

5. Give the value of the five smallest Fermat numbers.

5 bonus

Five bonus points for correctly describing the form a Fermat number has in binary.

Test 1 continued next page.

$25 + 8$

5

6. Form the best **match** among the following.

_____	*versus*	A.	make weight
_____	*mantissa*	B.	about
_____	*circa*	C.	against
_____	*modulo*	D.	that is
_____	*id est*	E.	a small measure

5

7. Explicitly indicate the prime factorization of 270 and 600. Be sure to use exponents and list the prime factors in increasing order.

5

____ **8**. Find LCM(270,600).

5

____ **9**. Convert 543_{10} into its base 6 value.
(**Three bonus points:** Convert 543_6 into its base 10 value.)

5

10. Depict a Pascal's triangle with sides of length 6. **Two bonus points** for naming the mathematical/calculator function which will give each entry directly. **Two more bonus points** for giving the formula for evaluating this mathematical function.

Test 1 continued next page.

$25 + 7$

5

11. From the conditional: "If no clouds, then no rain."; write the:

____________________ a. Converse

____________________ b. Inverse

____________________ c. Contrapositive

____________________ d. p

____________________ e. q.

10

12,13. You are given a three input logic gate whose output is described completely as the most common input. Fill in the missing two input and eight output values in the table below. **Four bonus points:** how can the output be described simply by considering separately $p = 0$ and $p = 1$?

p	q	r	most(p, q, r)
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1			

5

14. Solve for x and graph the solution set of $-2x + 9 < 1$.

5

15. Express the unit fraction $\frac{1}{13}$ as a decimal fraction **exactly**. How many digits are there in the portion which repeats? **Five bonus points** for identifying which multiples of $\frac{1}{13}$ can be represented by starting this repetition at a different point?

Test 1 continued next page.

$25 + 9$

5

16. Form the best **match** among the following.

_____ *Pauca, sed matura* A. Archimedes

_____ *Book of Nature is written in mathematical characters* B. Newton

_____ *Cogito ergo sum* C. Bernoulli

_____ *... playing on the seashore...smoother pebble* D. Galileo

_____ *Eureka, Eureka* E. Gauss

F. Descartes

10

____ **17,18**. Show work evaluating **by hand**: $\dfrac{\frac{1}{3}+\frac{1}{4}}{\frac{1}{5}+\frac{1}{2}}$

5

____ ____ **19**. Express the number representing the diagonal of a unit square in several forms (3 points). Be sure to include at least one with a fractional exponent (2 points).

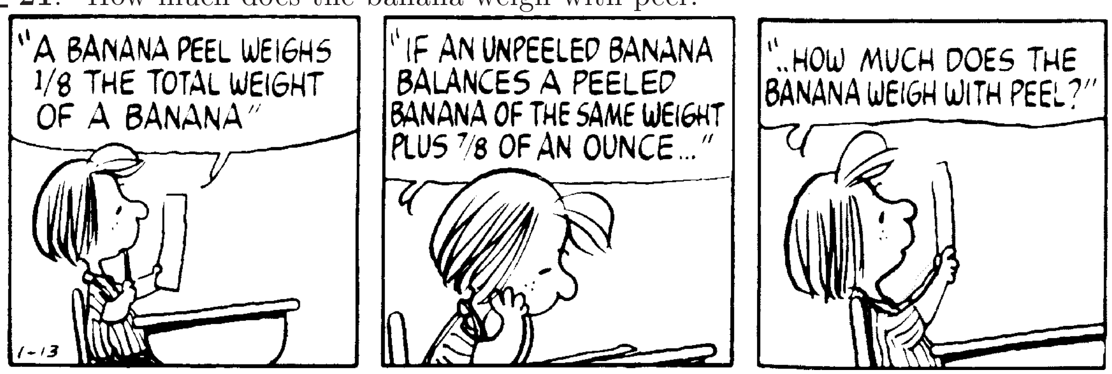
5

____ **20**. Rationalize the demoninator and simplify completely: $\sqrt{\frac{225}{18}}$.

Bonus Question, 5 bonus points

0

____ **21**. How much does the banana weigh with peel?



I have been careful to not allow others to see my work and the work on this examination is completely my own. This examination is returned and associated solutions are provided for my own personal use only. I may not share them except with concurrent classmates taking the identical course. Other uses are not condoned. I will dispose of it properly.

signature date

End of Test.—Check your work.—Have a nice day!

$25+5$

Name ______________________________ Score ________

C.2 Geometry, Test 1, September 24, 2004—Released Test

```
One 3"x5" notecards and TI-84+ type graphic calculator allowed.
Please place answers on the short underlines provided to the
left of the problem symbol.  Each of the 21 question numbers
has equal weight (i.e. 5 points each).  Question subparts have
about equal weight.  Read the questions carefully.  Hand in any
used scratch paper with the test for potential partial credit.
SHOW YOUR WORK
```

5

1. Form the best **match** among the following.

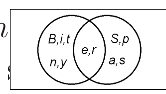
C	Triangular Numbers	A.	$0, 1, 4, 9, \ldots$
A	Squares	B.	$0, 1, 1, 2, \ldots$
D	Perfects	C.	$0, 1, 3, 6, \ldots$
E	The Factorials	D.	$6, 28, 496, \ldots$
B	The Fibonacci Numbers	E.	$1, 1, 2, 6, \ldots$

5

5+3

$\{e, r\}$ 2 pt each; 1 pt {} bonus: 7, 6, 2

5

___ **2**. Perform the following set operation: $\{B, r, i, t, n$  $p, e, a, r, s\}$.

(**Three bonus points:** what is the cardinality of each s

$\{B, r, i, t, n, e, y, S, p, a, s\}$

1 pt for box around VD

5

5 210

___ **3**. Perform the following set operation and sketch the corresponding Venn diagram. $\{B, r, i, t, n, e, y\} \cup \{S, p, e, a, r, s\}$.

$\frac{7!}{4!} = \frac{7 \cdot 6 \cdot 5 \cdot 4!}{4!} = 42 \cdot 5 = 210.$

5

___ **4**. Explicitly use the recursive definition of $n!$ to simplify then evaluate: $\frac{7!}{4!}$.

2 pts explicit cancellation of 4!, 1 pt ans

5

5. Give the value of the five smallest Fermat numbers.

$2^{2^0}+1,\ 2^{2^1}+1,\ 2^{2^2}+1,\ 2^{2^3}+1,\ 2^{2^4}+1,\ 2^{2^5}+1$

3, 5, 17, 257, 65537, 4294967297; an end option

5

5 bonus 5

Five bonus points for correctly describing the form a Fermat number has in binary. 11_2, 101_2, 10001_2, 100000001_2, *etc.*

Starts/ends with 1, has $2^n - 1$ zeroes in between

Test 1 continued next page.

33

25 + 8

5

[ ] 5

6. Form the <u>best</u> **match** among the following.

C			
A	*versus*	A.	make weight
B	*mantissa*	B.	about
E	*circa*	C.	against
D	*modulo*	D.	that is
____	*id est*	E.	a small measure

5

[ ] 5

7. Explicitly indicate the prime factorization of 270 and 600. Be sure to use exponents and list the prime factors in increasing order.

$270 = 2 \cdot 3^3 \cdot 5 \qquad 600 = 2^3 \cdot 3 \cdot 5^2$

Use TI-8x+ MATH NUM 8

5

[ ] 5

____ **8**. Find LCM(270,600).

$GCF(270, 600) = 30$

Also, $270 = 30 \cdot 9 \quad 600 = 30 \cdot 20$

$LCM = \frac{270 \cdot 600}{GCF(270,600)} = \frac{270 \cdot 600}{30} = 270 \cdot 20 = 5400$

so $5400 = 20 \cdot 270 = 9 \cdot 600 = 2^3 \cdot 3^3 \cdot 5^2$

____ **9**. Convert 543_{10} into its base 6 value.

5+3

[ ] 5

(**Three bonus points:** Convert 543_6 into its base 10 value.)

543/6=90R3; 90/6=15R0; 15/6=2R3; so $543_{10} = 2$

Chk: $2 \cdot 6^3 + 3 \cdot 6^2 + 3 \cdot 6^0 = 2 \cdot 216 + 3 \cdot 36 + 3 \cdot 1 = 432 + 108$

$543_6 = 5 \cdot 6^2 + 4 \cdot 6^1 + 3 \cdot 6^0 = 180 + 24 + 3 = 207$

$= 1000_6 - 13_6 = 216 - 9 = 207$

[ ] 5+4

10. Depict a Pascal's triangle with sides of length 6. **Two bonus points** for naming the mathematical/calculator function which will give each entry directly. **Two more bonus points** for giving the formula for evaluating this mathematical function.

Combination or ${}_nC_r$ or Choose/Choice

see MATH PRB 4 on TI-8x+

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

Test 1 continued next page.

$\boxed{32}$

25 + 7

5

If no rain, then no clouds.

5 **11**. From the conditional: "If no clouds, then no rain."; write the:

If clouds, then rain.

a. Converse

If rain, then clouds.

b. Inverse

no clouds (has if: $-\frac{1}{2}$ point)

c. Contrapositive

no rain (has then: $-\frac{1}{2}$ point)

d. p

e. q.

10+4

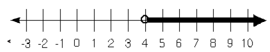
10 **12,13**. You are given a three input logic gate whose output is described completely as the most common input. Fill in the missing two input and eight output values in the table below. **Four bonus points:** how can the output be described simply by considering separately $p = 0$ and $p = 1$?

p	q	r	most(p, q, r)
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$p = 0 \Leftrightarrow q \wedge r$ $p = 1 \Leftrightarrow q \vee r$

p select gate type: $p = 0$ is and; $p = 1$ is o

5

5 **14**. Solve for x and graph the solution set of $-2x + 9$ 

3 pts: $-2x < -8$ $x > 4$

2 pts: direction, open circle

5+5

5 **15**. Express the unit fraction $\frac{1}{13}$ as a decimal fraction **exactly**. How many digits are there in the portion which repeats? **Five bonus points** for identifying which multiples of $\frac{1}{13}$ can be represented by starting this repetition at a different point?

$\frac{1}{13} = 0.\overline{076923}$ 6 digits repeat.

$\{1, 3, 4, 9, 10, 12\}$ see also $\{2, 5, 6, 7, 8, 11\}$

Note the symmetry in these groups.

Test 1 continued next page.

34

$25 + 9$

5

16. Form the best **match** among the following.

_____	*Pauca, sed matura*	A. Archimedes
_____	*Book of Nature is written in mathematical characters*	B. Newton
_____	*Cogito ergo sum*	C. Bernoulli
_____	*... playing on the seashore...smoother pebble*	D. Galileo
_____	*Eureka, Eureka*	E. Gauss
		F. Descartes

10

____ **17,18**. Show work evaluating **by hand**: $\dfrac{\frac{1}{3}+\frac{1}{4}}{\frac{1}{5}+\frac{1}{2}}$

5

____ ____ **19**. Express the number representing the diagonal of a unit square in several forms (3 points). Be sure to include at least one with a fractional exponent (2 points).

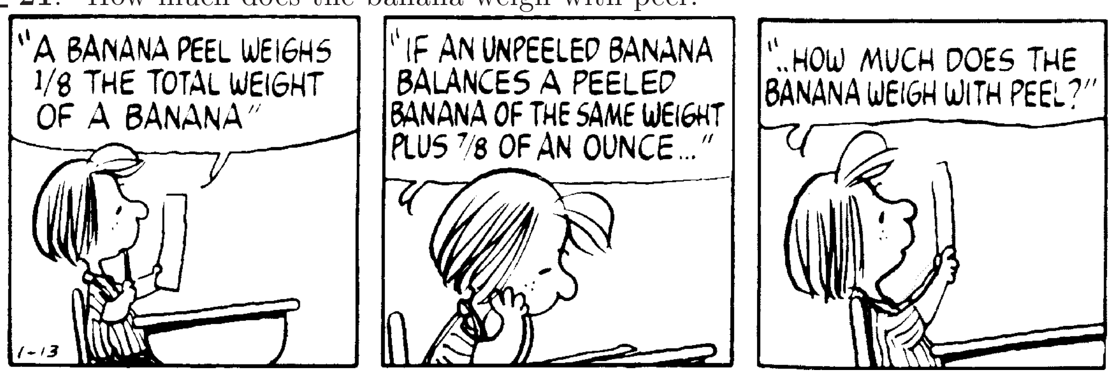
5

____ **20**. Rationalize the demoninator and simplify completely: $\sqrt{\frac{225}{18}}$.

Bonus Question, 5 bonus points

0

____ **21**. How much does the banana weigh with peel?



I have been careful to not allow others to see my work and the work on this examination is completely my own. This examination is returned and associated solutions are provided for my own personal use only. I may not share them except with concurrent classmates taking the identical course. Other uses are not condoned. I will dispose of it properly.

_______________________________ signature   __________ date

End of Test.—Check your work.—Have a nice day!

$25+5$