Rocky Mountain Mathematics Consortium
Summer Conference
University of Wyoming
7-18 July 2003

on

Discrete Dynamical Systems
& Applications to Population Dynamics

Principle Speakers

J. M. Cushing
U of Arizona

Shandelle M. Henson
Andrews University

First week
Discrete Dynamical Systems

Second week
Applications and Case Studies

Special Speakers

R. F. Costantino
U of Arizona

Aaron King
U of Tennessee

Brian Dennis
U of Idaho

First Week
Discrete Dynamical Systems

Lecture 1: Introduction (JC)
Lecture 2: Linear Maps (Henson)
Lecture 3: Linear/Nonnegative Matrix Models (JC)
Lecture 4: Nonlinear, Autonomous Maps (Henson)
Lecture 5: Local bifurcations (Henson)
Lecture 6: Nonlinear Matrix Models (JC)
Lecture 7: Periodically forced maps (Henson)
Lecture 8: Topics in Chaos I (JC)
Lecture 9: Topics in Chaos II (JC)
Lecture 10: Topics in Chaos III(?)
and/or Multi-species Models (JC)

Second Week
Studies in Population Dynamics & Ecology

Lecture 1: Mathematics & Biology (JC & Costantino)
Lecture 2: The LPA Model (JC)
Lecture 3: Connecting Models to Data I (Dennis)
Lecture 4: Connecting Models to Data II (Dennis)
Lecture 5: Chaos I (Costantino)
Lecture 6: Chaos II (King)
Lecture 7: Patterns in Chaos (King)
Lecture 8: Competing Species (JC)
Lecture 9: Periodic Habitats (Henson)
Lecture 10: Periodic Habitats (Henson)

First Week Themes

- Asymptotic dynamics
- Stability theory
- Bifurcations
- Chaos

Supplementary reading

Elementary level

- James T. Sandefur, "Discrete Dynamical Systems: Theory & Applications"

Undergraduate level

- Saber N. Elaydi, "Discrete Chaos"
- Richard A. Holmgren, "A First Course in Discrete Dynamical Systems"

Undergraduate/Graduate level

- Saber N. Elaydi, "An Introduction to Difference Equations"
- H. Caswell, "Matrix Population Models: Construction, Analysis and Interpretation", Second edition

Graduate level

- S. Wiggins, "Intro to Applied Nonlinear Dynamical Systems and Chaos"
- Guckenheimer & Holmes, "Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields"

Lecture #1

$x =$ vector of state variables

$t =$ time

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in R^m = R \times \cdots \times R \text{ where } R = \text{reals}$$

$$t \in Z = \{\cdots, -2, -1, 0, 1, 2, \cdots\} \text{ or } Z_+ = \{0, 1, 2, \cdots\}$$

$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{pmatrix}$$

A Preliminary Example

$$x(t+1) = a\,x(t)$$

$$x \in R, \quad a \in R, \quad t \in Z$$

Recursive formula

Difference equation

$$x(t) = a^t x_0 \quad \text{where} \quad x_0 = x(0)$$

For each $t \in Z$ define

$$T^t : R \to R$$

by $x_0 \to a^t x_0$

NOTE :

(1) $T^0 =$ identity map (obvious)

(2) $T^{p+q} = T^p \circ T^q$ (semi - group property)

$$T^{p+q}\left(x_0\right) = a^{p+q} x_0 = a^p \left(a^q x_0\right) = T^p\left(T^q\left(x_0\right)\right)$$

<u>DEFINITION</u> : Let X be a set. A <u>metric</u> is a function

$$d : X \times X \to R^1_+$$

that satisfies the three conditions :

$$d(x,y) = 0 \Leftrightarrow x = y$$
$$d(x,y) = d(y,x) \ \forall \ x,y \in X$$
$$d(x,y) \le d(x,z) + d(z,y) \ \forall \ x,y,z \in X$$

X is called a <u>metric space.</u>

DEFINITION: A <u>discrete dynamical system</u> is a one parameter family of continuous maps

$$T^t : X \to X = \text{metric space}, \quad t \in Z$$

satisfying

$$T^0 = \text{identity map}$$

$$T^{p+q} = T^p \circ T^q \text{ for all } p,q \in Z.$$

Replace Z by Z_+ and T^t is a discrete <u>semi-dynamical</u> system.

A difference equation (recursive formula)

$$x(t+1) = f\left(x(t)\right)$$

$$x \in D, \quad t \in Z \quad \left(\text{or } t \in Z_+\right)$$

$$f : D \xrightarrow{\text{conts}} D = \text{subset of a metric space}$$

defines discrete dynamical system on $X = D$

(or a discrete semi-dynamical system)

Two Basic Classifications

(1) Dimension: A map

$$f : D \to D = \text{open} \subseteq R^m$$

defines an m - dimensional dynamical system on D.

EXAMPLE

$$f(x) = b\frac{1}{1+cx}x,\ \ b>0,\ \ c \ge 0$$

$$f : R_+ \to R_+ = \{0 \le r \in R\}$$

defines an one dimensional dynamical system on R_+

$$x(t+1) = b\frac{1}{1+cx(t)}x(t), \quad t \in Z_+$$

EXAMPLE

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} b_1 \dfrac{1}{1+c_{11}x_1 + c_{12}x_2} x_1 \\ b_2 \dfrac{1}{1+c_{21}x_1 + c_{22}x_2} x_2 \end{pmatrix},\ \ b_i > 0,\ \ c_{ij} \ge 0$$

$$f : R_+^2 \to R_+^2 = R_+ \times R_+$$

defines an two dimensional dynamical system

$$x_1(t+1) = b_1 \frac{1}{1+c_{11}x_1(t) + c_{12}x_2(t)} x_1(t)$$

$$x_1(t+1) = b_2 \frac{1}{1+c_{21}x_1(t) + c_{22}x_2(t)} x_2(t)$$

(2) Linearity: A map $f : R^m \to R^m$ of the form

$$f(x) = Ax + h$$

$$h \in R^m, \quad A = m \times m \text{ matrix}$$

defines an m-dimensional linear system.

The system is homogeneous if $h = 0$.

EXAMPLE

$$x(t+1) = Lx(t)$$

$$L = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ \tau_{21} & 0 & 0 & 0 \\ 0 & \tau_{33} & 0 & 0 \\ 0 & 0 & \tau_{43} & \tau_{44} \end{pmatrix}$$

$$b_i \ge 0,\ \ \tau_{ij} \ge 0$$

A four dimensional system on the "non-negative cone" R_+^4

The formal theory of dynamical systems is a powerful and very general theory.

As we have seen "autonomous" difference equations

$$x(t+1) = f(x(t))$$

define discrete dynamical systems.

It is not always so obvious how to formulate non-autonomous difference equations

$$x(t+1) = f(t, x(t))$$

into the general theory in a useful way.

Rewrite the m-dimensional, non-autonomous problem

$$x(t+1) = f(t, x(t))$$
$$x(0) = x_0$$

as the $(m+1)$-dimensional autonomous problem

$$x(t+1) = f(y(t), x(t))$$
$$y(t+1) = y(t) + 1$$
$$x(0) = x_0, \quad y(0) = 0$$

Difficulty: all orbits of this autonomous problem are unbounded.

To formulate a problem as a dynamical system one typically takes advantage of special features of the equations (e.g., periodicity).

We will focus on differences equations per se.

ASYMPTOTIC DYNAMICS

Some Basic Definitions

An (open) ball: $B(x,r) = \{y \in X : d(x,y) < r\}$

A set $S \subset X$ is "open" if for each $x \in S \ \exists \ B(x,r) \subset S$

$x \in X$ is a "limit point" of $S \subset X$
if $\exists \ y_n \in S$ such that $\lim_{n\to\infty} d(x, y_n) = 0$

The "closure" $\bar{S}$ of $S \subset X$ is $S \cup \{\text{all limit points of } S\}$

$S \subset X$ is "closed" if $S = \bar{S}$

$S \subset X$ is "dense in X" if $\bar{S} = X$

$O(x_0) = \{ T^t x_0 : t \in Z \}$ = the orbit through x_0

$O_+(x_0) = \{ T^t x_0 : t \in Z_+ \}$ = the forward orbit through x_0

A limit point of a forward orbit $O_+(x_0)$ is the omega limit point of the orbit.

$\omega(x_0) =$ {omega limit points of $O_+(x_0)$} is the omega limit set of the orbit.

Basic Properties of Omega Limit Sets of bounded orbits on R^m

(1) $\omega(x_0) \neq \varnothing$

(2) $\omega(x_0)$ is bounded

(3) $\omega(x_0)$ is closed

(4) $\omega(x_0)$ is forward invariant

(forward orbits of omega limit points remain in $\omega(x_0)$)

EXAMPLES

$$x(t+1) = a\,x(t), \quad x(0) = x_0$$

Forward orbits: $O_+(x_0) = \left\{ a^t x_0 : t \in Z_+ \right\}$

(1) $a = \frac{1}{2} \Rightarrow O_+(1) = \left\{ \left(\frac{1}{2}\right)^t : t \in Z_+ \right\} \Rightarrow \omega(1) = \{0\}$

Note: $O(0) = \{0\}$ and $f(\{0\}) = \{0\}$

(2) $a = -1 \Rightarrow O_+(1) = \left\{ (-1)^t : t \in Z_+ \right\} \Rightarrow \omega(1) = \{1, -1\}$

Note: $f(\{-1,1\}) = \{-1,1\}$

DEFINITION

A constant solution (point orbit) is called an equilibrium.

Equilibria are fixed points of f : $f(x) = x$

DEFINITION

A solution satisfying $x(t+p) = x(t)$ for all t and a (smallest) integer $p \geq 1$ is called a p-cycle

p-cycles are determined by the fixed points of the composite map

$$f^{(p)}(x) \doteq f\left(f\left(\cdots f(x)\right)\right) = x$$

DEFINITION:

A set $A \subset X$ (a metric space) is an attractor if

(a) $f(A) = A$

(b) there is an open set $U \supset A$ such that

$$x_0 \in U \Rightarrow \omega(x_0) \subset A$$

(c) no subset of A has property (a)

A is a global attractor if $U = X$

EXAMPLES

$$x(t+1) = ax(t), \quad x(0) = x_0$$

(1) $a = \frac{1}{2} \Rightarrow O_+(x_0) = \left\{ \left(\frac{1}{2} \right)^t x_0 : t \in Z_+ \right\} \Rightarrow \omega(x_0) = \{0\}$

$\Rightarrow A = \{0\}$ (the equilibrium) is a global attractor

(2) $a = -1 \Rightarrow O_+(1) = \left\{ (-1)^t : t \in Z_+ \right\} \Rightarrow \omega(1) = \{1, -1\}$

But $A = \{1, -1\}$ is not an attractor.

$$f^{(2)}(x) \doteq f\left(f(x)\right) = -(-x) = x$$

$\Rightarrow$ all points (except $x = 0$) are 2 - periodic points.

$$O_+(x_0) = \left\{ (-1)^t x_0 : t \in Z_+ \right\} \Rightarrow \omega(x_0) = \{x_0, -x_0\}$$

None of the 2-cycles is an attractor

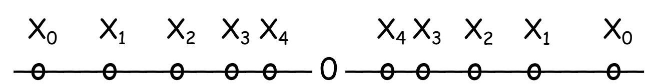
$$x(t+1) = ax(t), \quad x(0) = x_0$$

Forward orbits : $O_+(x_0) = \left\{ a^t x_0 \right\}$

$0 < a < 1 \Rightarrow$ the equilibrium $x = 0$

is a global attractor

Note : all solutions are monotonic.



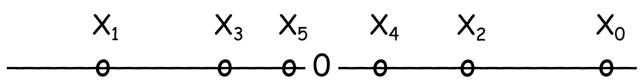
$$x(t+1) = a\,x(t), \quad x(0) = x_0$$

Forward orbits: $O_+(x_0) = \{a^t x_0\}$

$-1 < a < 0 \Rightarrow$ the equilibrium $x = 0$

is a global attractor

Note : all solutions are oscillatory.



Finally …

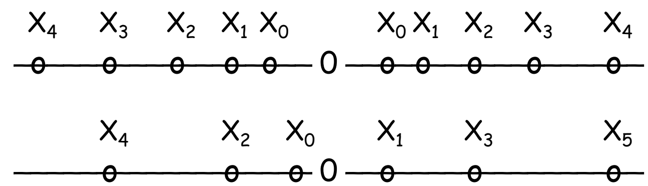
$1 < |a| \Rightarrow \lim_{t\to\infty} |x(t)| = \infty$ for all $x_0 \in R$

All solutions are unbounded.

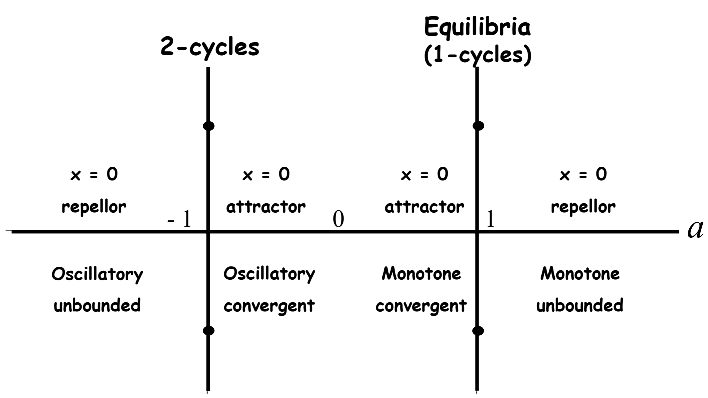
If $1 < a$, the solutions are monotonic.

If $a <$ -1, the solutions are oscillatory.

In either case, the equilibrium $x = 0$ is a repellor.



Graphical Summary: Bifurcation Diagram



a = -1 and 1 are called bifurcation points

A Linear Application

$x(t) =$ population numbers or density at time t

$x(t+1) =$ recruitment (births) + survivors

$$x(t+1) = bx(t) + dx(t)$$

$0 \le b$ = per capita recruitment rate per unit time

$0 \le d$ = fraction that survive a unit of time

$$x(t+1) = ax(t), \quad a = b + d \ge 0$$

$0 \le a < 1 \Rightarrow x = 0$ is a global attractor (extinction)

$1 < a \Rightarrow x = 0$ is a repellor (unbounded growth)

$a = 1 \Rightarrow x(t) =$ equilibrium (bounded survival)

$$\text{Define}: \quad n = \frac{b}{1-d} \ge 0$$

$0 \le n < 1 \Rightarrow x = 0$ is a global attractor (extinction)

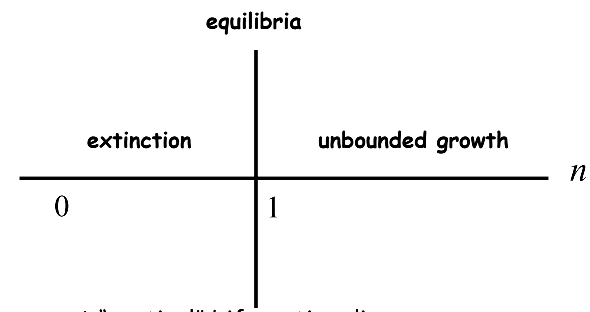
$1 < n \Rightarrow x = 0$ is a repellor (unbounded growth)

$n = 1 \Rightarrow x(t) = x_e$ equilibrium (bounded survival)

$$n = \frac{b}{1-d} = b + bd + bd^2 + bd^3 + \cdots$$

expected per capita number of recruits per lifetime

or the <u>net reproductive number</u>



A "vertical" bifurcation diagram.

Vertical bifurcations (point spectra) are typical of linear equations

Most applications involve nonlinear equations

<u>Methods of analysis</u>

- ✓ Solution formulas (rare)
- ✓ Geometric analysis (dimensionally limited)
- ✓ "Qualitative" analysis

The following material was not covered in Lecture #1

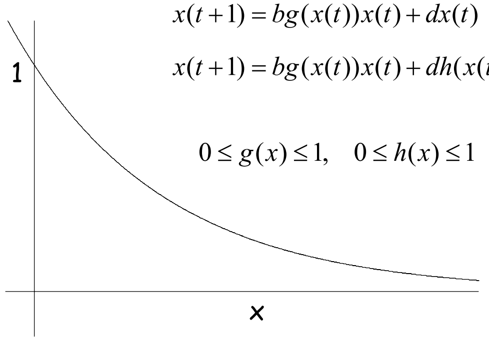
A Nonlinear Application

Regulated Population Growth

$$x(t+1) = bx(t) + dx(t)$$

$$x(t+1) = bg(x(t))x(t) + dx(t)$$

$$x(t+1) = bg(x(t))x(t) + dh(x(t))x(t)$$



$$0 \le g(x) \le 1, \quad 0 \le h(x) \le 1$$

Example

(Beverton/Holt)

$d = 0$ (non-overlapping generations)

$$g(x) = \frac{1}{1+cx}, \quad c > 0$$

$$x(t+1) = b\frac{1}{1+cx(t)}x(t), \quad b > 0$$

$$x(0) = x_0 \ge 0$$

What are the asymptotic dynamics?

Approach #1

(Solution Formula)

If $b < 1$, then $0 \le x(t+1) \le bx(t)$

$$\Rightarrow 0 \le x(t) \le b^t x_0 \to 0.$$

Assume $b > 1$.

Change variable: $y(t) = \dfrac{1}{x(t)}$.

Beverton/Holt equation transforms into a linear equation for $y(t)$.

$$y(t+1) = b^{-1}y(t) + b^{-1}c$$

By induction (a homework problem):

$$y(t) = b^{-t}y_0 + \frac{c}{b-1}\left(1 - b^{-t-1}\right)$$

$$x(t) = \frac{(b-1)x_0}{(b-1)b^{-t} + x_0 c\left(1 - b^{-t-1}\right)}$$

$$\Rightarrow \lim_{t\to\infty} x(t) = (b-1)/c$$

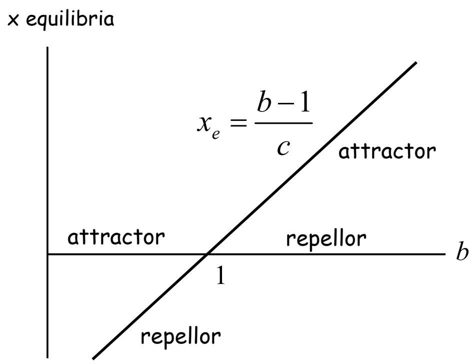
Homework problem:

Show $x_e = (b-1)/c$ is an equilibrium.

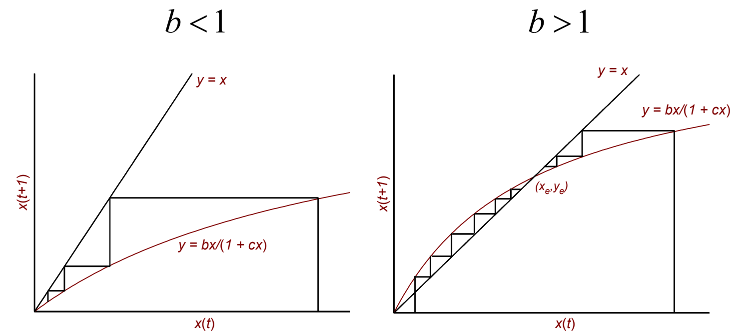
Summary

$$b < 1 \Rightarrow x(t) \to 0 \quad \text{(extinction)}$$

$$b > 1 \Rightarrow x(t) \to x_e = \text{equilibrium} \quad \text{(bounded survival)}$$



Approach #2
(a geometric approach)



Approach #3
(analytic)

If $b < 1$, then $0 \le x(t+1) \le bx(t)$

$$\Rightarrow 0 \le x(t) \le b^t x_0 \to 0.$$

Assume $b > 1$ and $x_0 > 0$.

$$0 < b\frac{1}{1+cx}x < b\frac{1}{c} \quad \text{for all } x > 0$$

$\Rightarrow$ for $x_0 > 0$ after one step $0 < x(t) < b/c$

$\Rightarrow$ all solutions are nonnegative and bounded above and below (by 0).

$f(x) = b\dfrac{1}{1+cx}x$ is monotone increasing in x.

$$0 < x_0 < x_e$$

$$\Rightarrow f(x_0) < f(x_e) = x_e$$

$\Rightarrow x(1) < x_e \Rightarrow x(t) < x_e$ for all t (by induction)

$$0 < x_0 < x_e$$

$$\Rightarrow x_0 < (b-1)/c \Rightarrow 1 < \frac{b}{1+cx_0}$$

$$\Rightarrow x_0 < \frac{b}{1+cx_0}x_0 = x(1)$$

$\Rightarrow x(t)$ is monotone increasing (by induction)

Similar kinds of arguments show

$x_e < x_0 \Rightarrow x_e < x(t)$ is monotone decreasing

CONCLUSION: all solutions are monotonic and are bounded above and below.

Thus, all solutions converge to a limit.

$$\lim_{t\to\infty} x(t) = x_\infty \geq 0.$$

$$\lim_{t\to\infty} x(t+1) = \lim_{t\to\infty} b\frac{1}{1+cx(t)}x(t)$$

$$x_\infty = b\frac{1}{1+cx_\infty}x_\infty$$

A little algebra shows $x_\infty = 0$ or $x_e = (b-1)/c$

For $b > 1$ we showed

$0 < x_0 < x_e \Rightarrow x(t) < x_e$ is increasing

$0 < x_e < x_0 \Rightarrow x_e < x(t)$ is decreasing

therefore

$$0 < x_0 < x_e \Rightarrow x(t) \to x_e$$

$$0 < x_0 < x_e \Rightarrow x(t) \to x_e$$