RELATIONSHIPS WITHIN TRIANGLES

Geometry

Chapter 6

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- This Slideshow was developed to accompany the textbook
  - *Big Ideas Geometry*
  - *By Larson and Boswell*
  - *2022 K12 (National Geographic/Cengage)*
- Some examples and diagrams are taken from the textbook.

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6.1 PERPENDICULAR AND ANGLE BISECTORS

After this lesson…

- I can identify a perpendicular bisector and an angle bisector.
- I can use theorems about bisectors to find measures in figures.
- I can write equations of perpendicular bisectors.

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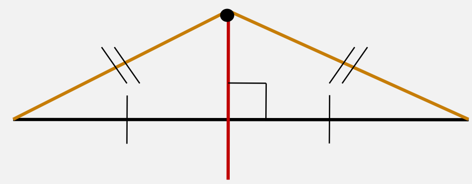
6.1 PERPENDICULAR AND ANGLE BISECTORS

- **Perpendicular Bisector**
  - Segment that is perpendicular to and bisects a segment



Perpendicular Bisector Theorem

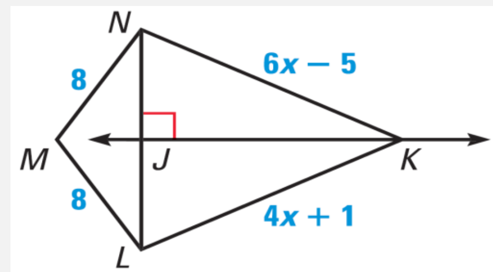
If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment

Converse of the Perpendicular Bisector Theorem

If a point is equidistant from the endpoints of a segment, then it is on the perpendicular bisector of the segment

6.1 PERPENDICULAR AND ANGLE BISECTORS

- In the diagram, $\overleftrightarrow{JK}$ is the perpendicular bisector of $\overline{NL}$.
- Find NK.



- Explain why M is on $\overleftrightarrow{JK}$.
- Try #6

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Since JK is $\perp$ bisector, then NK = LK ($\perp$ bisector theorem).

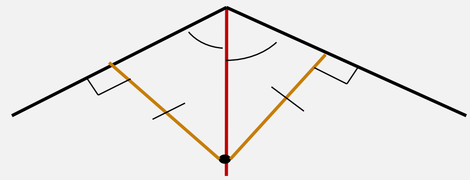
$$6x - 5 = 4x + 1 \rightarrow 2x - 5 = 1 \rightarrow 2x = 6 \rightarrow x = 3$$

Find NK: $6x - 5 \rightarrow 6(3) - 5 = 13$

Since MN = ML, M is equidistant from each end of NL. Thus by then Converse of the Perpendicular Bisector Theorem, M is on the perpendicular bisector.

6.1 PERPENDICULAR AND ANGLE BISECTORS

- **Angle Bisector**
  - Ray that bisects an angle



Angle Bisector Theorem

If a point is on the angle bisector, then it is equidistant from the sides of the angle

Converse of the Angle Bisector Theorem

If a point is equidistant from the sides of an angle, then it is on the angle bisector

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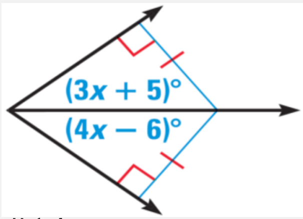
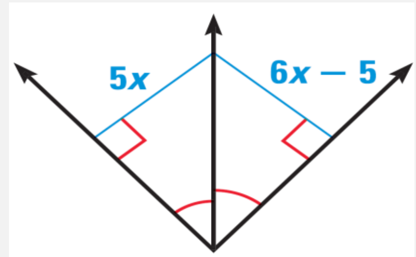
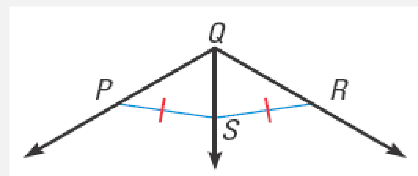
6.1 PERPENDICULAR AND ANGLE BISECTORS

- Find the value of x.





- Try #14
- Do you have enough information to conclude that $\overrightarrow{QS}$ bisects ∠PQR?



- Try #10

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$$3x + 5 = 4x - 6 \rightarrow 5 = x - 6 \rightarrow x = 11$$

$$5x = 6x - 5 \rightarrow -x = -5 \rightarrow x = 5$$

No, you need to know that SP ⊥ QP and SR ⊥ QR

6.1 PERPENDICULAR AND ANGLE BISECTORS

- **Write Equations of Perpendicular Bisector**

1. Find midpoint
2. Find slope
3. Find ⊥ slope
4. Write equation using slope from #3 and point from #1

- Write the perpendicular bisector of a segment with endpoints $D(5, -1)$ and $E(-11, 3)$

- Try #20

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1. Midpoint: $M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \to \left(\frac{5+(-11)}{2}, \frac{-1+3}{2}\right) \to (-3, 1)$
2. Slope: $m = \frac{y_2-y_1}{x_2-x_1} \to \frac{3-(-1)}{-11-5} = \frac{4}{-16} = -\frac{1}{4}$
3. ⊥ Slope: $m = 4$
4. Equation:

$$y = mx + b$$
$$1 = 4(-3) + b$$
$$1 = -12 + b$$
$$13 = b$$
$$y = 4x + 13$$

6.2 BISECTORS OF TRIANGLES

After this lesson…

- I can find the circumcenter and incenter of a triangle.
- I can use points of concurrency to solve real-life problems.

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6.2 BISECTORS OF TRIANGLES

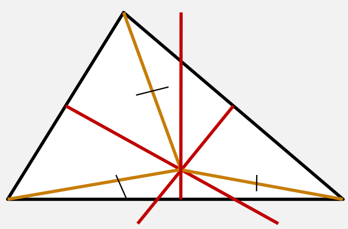
- Find the perpendicular bisectors of a triangle
- Cut out a triangle
- Fold each vertex to each other vertex
  - The three folds are the perpendicular bisectors
- What do you notice?
  - Perpendicular bisectors meet at one point
- Measure the distance from the meeting point to each vertex
- What do you notice?
  - The distances are equal

6.2 BISECTORS OF TRIANGLES

- **Concurrent**
  - Several lines that intersect at same point (point of concurrency)

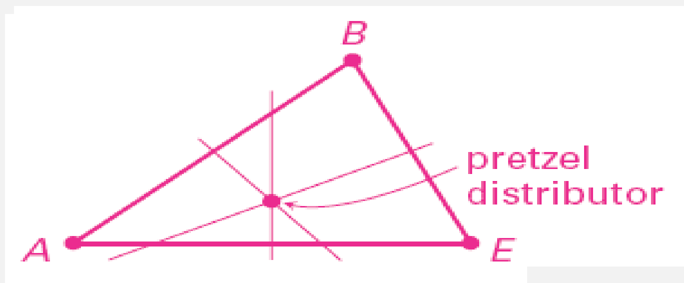
Concurrency of Perpendicular Bisectors of a Triangle

The perpendicular bisectors of a triangle intersect at a point that is equidistant from the vertices of a triangle



6.2 BISECTORS OF TRIANGLES

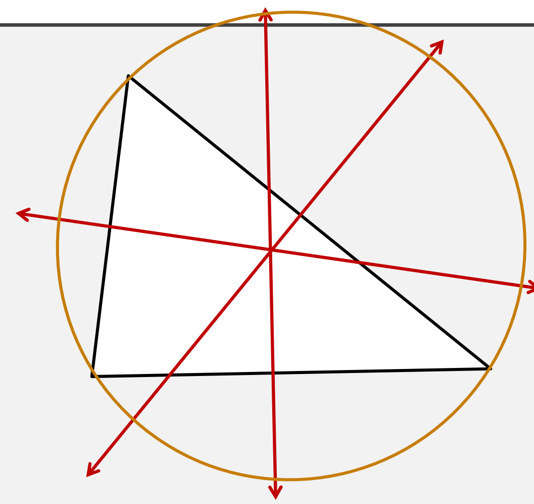
- Hot pretzels are sold from store at *A*, *B*, and *E*. Where could the pretzel distributor be located if it is equidistant from those three points?



- Try #1, 4

6.2 BISECTORS OF TRIANGLES

- **Circumcenter**
  - The point of concurrency of the perpendicular bisectors of a triangle.
  - If a circle was circumscribed around a triangle, the circumcenter would also be the center of the circle.

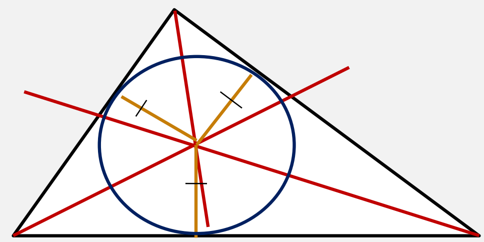


6.2 BISECTORS OF TRIANGLES

Concurrency of Angle Bisectors of a Triangle

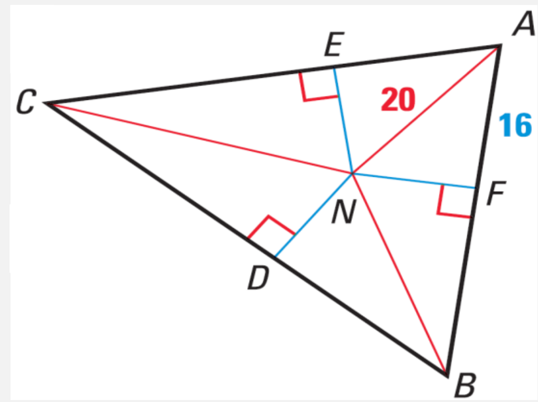
The angle bisectors of a triangle intersect at a point that is equidistant from the sides of a triangle

- **Incenter**
  - Point of concurrency of the angle bisectors of a triangle
  - If a circle was inscribed in a triangle, the incenter would also be the center of the circle.



6.2 BISECTORS OF TRIANGLES

- N is the incenter. Find EN.



- Try #6, 12

Find NF by using the Pythagorean theorem.

$$16^2 + NF^2 = 20^2 \rightarrow 256 + NF^2 = 400 \rightarrow NF^2 = 144 \rightarrow NF = 12$$

Since N is the incenter, NF = EN = 12

6.3 MEDIANS AND ALTITUDES OF TRIANGLES

After this lesson…

- I can find the centroid of a triangle.
- I can find the orthocenter of a triangle.

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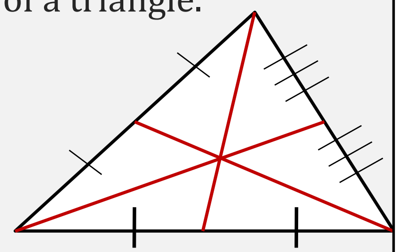
6.3 MEDIANS AND ALTITUDES OF TRIANGLES

- **Median**
  - Segment that connects a vertex to a midpoint of side of a triangle.
  - Point of concurrency is called the centroid.
  - The centroid is the balance point.

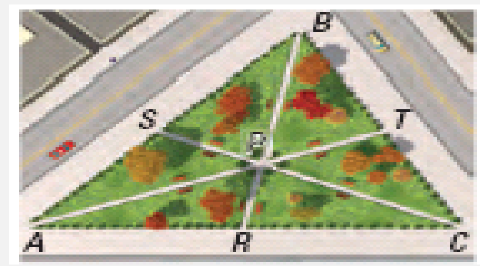


Concurrency of Medians of a Triangle

The medians of a triangle intersect at a point that is two thirds of the distance from each vertex to the midpoints of the opposite side.

6.3 MEDIANS AND ALTITUDES OF TRIANGLES

- Each path goes from the midpoint of one edge to the opposite corner. The paths meet at *P*.
- If *SC* = 2100 ft, find *PS* and *PC*.
- If *BT* = 1000 ft, find *TC* and *BC*.
- If *PT* = 800 ft, find *PA* and *TA*.



- Try #2, 6

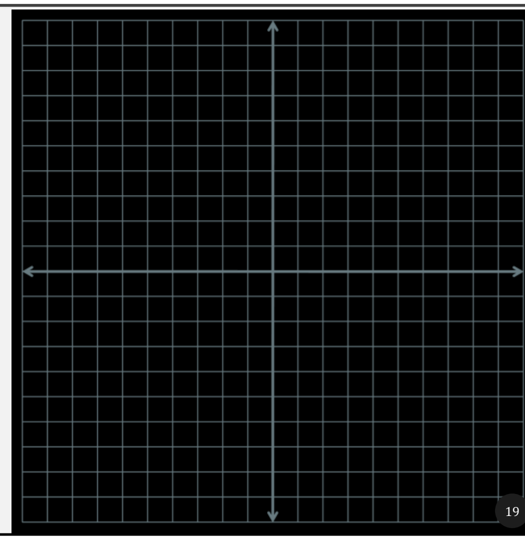
$$PC = \frac{2}{3}SC \rightarrow PC = \frac{2}{3}(2100) = 1400\ ft \rightarrow PS = 700\ ft$$

$$T\ is\ midpoint\ of\ BC.\ TC = 1000\ ft, BC = 2000\ ft$$

$$PT = \frac{1}{3}TA \rightarrow 800 = \frac{1}{3}TA \rightarrow 2400\ ft = TA, PA = 1600\ ft$$

6.3 MEDIANS AND ALTITUDES OF TRIANGLES

- Find the coordinates of the centroid of $\triangle ABC$ with vertices $A(0, 4)$, $B(-4, -2)$, and $C(7, 1)$.



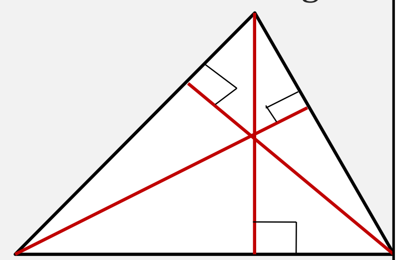
- Try #16

1. Graph the points
2. Find the midpoints of each side
   a. $M_{AB} = \left(\frac{0+(-4)}{2}, \frac{4+(-2)}{2}\right) = (-2, 1)$
   b. $M_{BC} = \left(\frac{-4+7}{2}, \frac{-2+1}{2}\right) = \left(\frac{3}{2}, -\frac{1}{2}\right)$
   c. $M_{AC} = \left(\frac{0+7}{2}, \frac{4+1}{2}\right) = \left(\frac{7}{2}, \frac{5}{2}\right)$
3. Connect each midpoint with the opposite vertex
4. The centroid is the intersection

(1, 1)

6.3 MEDIANS AND ALTITUDES OF TRIANGLES

- **Altitudes**
  - Segment from a vertex and perpendicular to the opposite side of a triangle.
  - Point of concurrency is called the orthocenter.



Concurrency of Altitudes of a Triangle

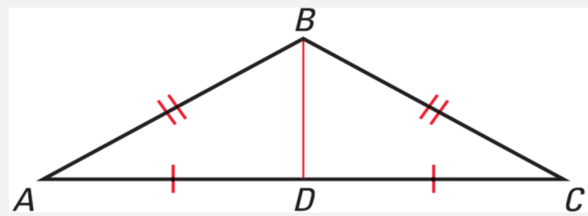
The lines containing the altitudes of a triangle are concurrent.

  - Acute Δ → orthocenter **inside** triangle
  - Right Δ → orthocenter **on right angle** of triangle
  - Obtuse Δ → orthocenter **outside** of triangle

There is nothing terribly interesting about the orthocenter.
In an acute triangle, the orthocenter is inside the triangle.
In a right triangle, the orthocenter is on the triangle at the right angle.
In an obtuse triangle, the orthocenter is outside of the triangle.

6.3 MEDIANS AND ALTITUDES OF TRIANGLES

- Find the orthocenter.

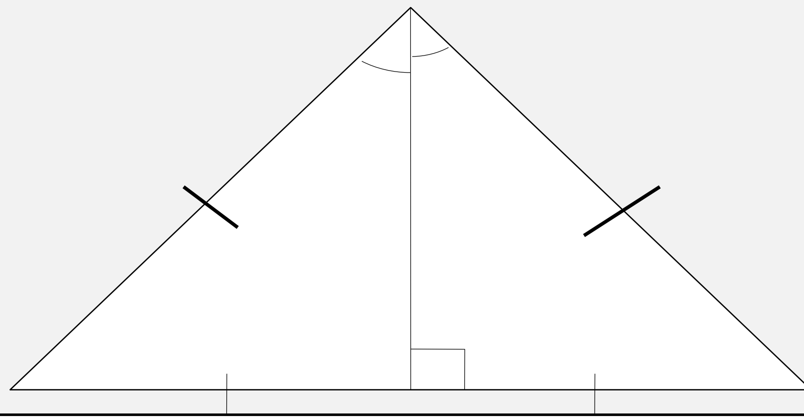


- Try #18

Draw the other two altitudes (from A and C). They will be outside the triangle

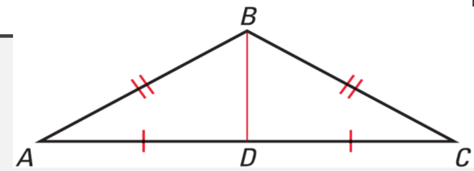
6.3 MEDIANS AND ALTITUDES OF TRIANGLES

- In an isosceles triangle, the perpendicular bisector, angle bisector, median, and altitude from the vertex angle are all the same segment.



6.3 MEDIANS AND ALTITUDES OF TRIANGLES

- Given: ΔABC is isosceles, $\overline{BD}$ is a median
- Prove: $\overline{BD}$ is an angle bisector



Statements	Reasons

Try #26

ΔABC is isosceles, BD is a median	(given)
BA ≅ BC	(def. Isosceles)
AD ≅ DC	(def. Median)
BD ≅ BD	(reflexive)
ΔABD ≅ ΔCBD	(SSS)
∠ABD ≅ ∠CBD	(def ≅ Δ) (CPCTC)
BD is an angle bisector	(def angle bisector)

6.4 THE TRIANGLE MIDSEGMENT THEOREM

After this lesson…

- I can use midsegments of triangles in the coordinate plane to solve problems.
- I can solve real-life problems involving midsegments.

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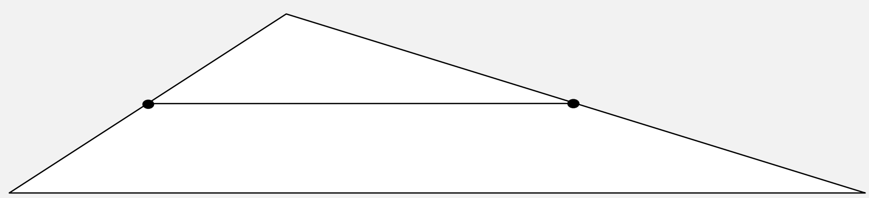
6.4 THE TRIANGLE MIDSEGMENT THEOREM

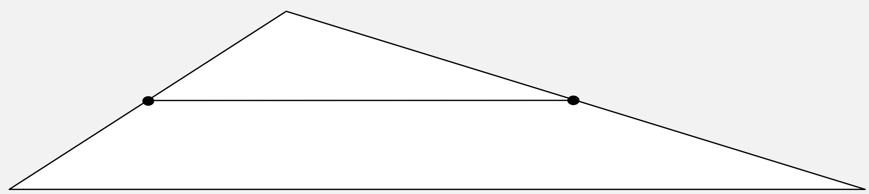
- Draw a triangle in your notes
- Find the midpoints of two of the sides using a ruler
- Connect the midpoints of the two sides with a segment
- Measure the segment and the third side
- What do you notice?
- What else do you notice about those two segments?



Length should be ½
They should be parallel

6.4 THE TRIANGLE MIDSEGMENT THEOREM

- **Midsegment of a Triangle**
  - Segment that connects the midpoints of two sides of a triangle

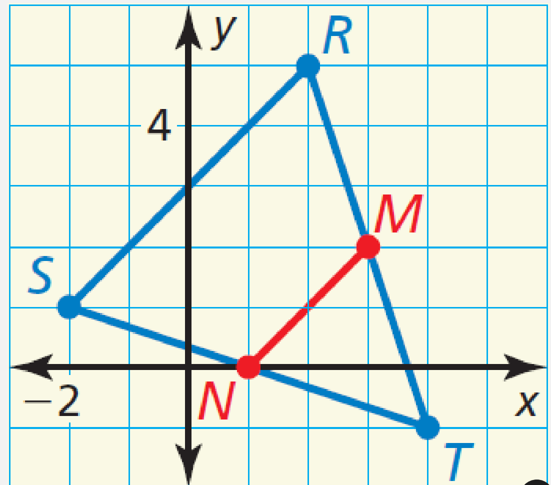


Midsegment Theorem

The midsegment of a triangle is parallel to the third side and is half as long as that side.

6.4 THE TRIANGLE MIDSEGMENT THEOREM

- In $\triangle RST$, show that midsegment $\overline{MN}$ is parallel to $\overline{RS}$ and that $MN = \frac{1}{2}RS$.



- Try #2

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Parallel (slopes):

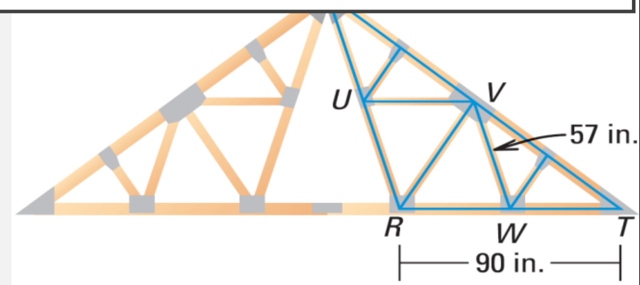
$$m_{MN} = \frac{2-0}{3-1} = 1$$
$$m_{RS} = \frac{5-1}{2-(-2)} = 1$$

Distance:

$$MN = \sqrt{(3-1)^2 + (2-0)^2} = 2\sqrt{2}$$
$$RS = \sqrt{\left(2-(-2)\right)^2 + (5-1)^2} = 4\sqrt{2}$$

6.4 THE TRIANGLE MIDSEGMENT THEOREM

- Name the midsegments.



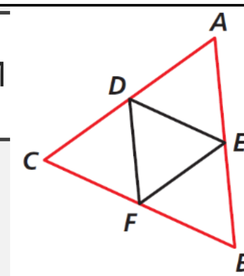
- Draw the third midsegment.
- Let UW be 81 inches. Find VS.
- Try #8, 10

UV, WV

UW

UW = ½ ST
VT = ½ ST
UW = VT = 81

6.4 THE TRIANGLE MIDSEGMENT THEOREM



- Given: $CF = FB$ and $CD = DA$
- Prove: $\overline{DF} \parallel \overline{AB}$

Statements	Reasons

Try #6

CF = FB, CD = DA	(given)
F is midpoint of CB, D is midpoint of AC	(def. Midpoint)
DF is midsegment	(def. Midsegment)
DF ǁ AB	(Midsegment Theorem)

6.5 INDIRECT PROOF AND INEQUALITIES IN ONE TRIANGLE

After this lesson…

- I can write indirect proofs.
- I can order the angles of a triangle given the side lengths.
- I can order the side lengths of a triangle given the angle measures.
- I can determine possible side lengths of triangles.

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6.5 INDIRECT PROOF AND INEQUALITIES IN ONE TRIANGLE

- Indirect Reasoning
  - You are taking a multiple choice test.
  - You don't know the correct answer.
  - You eliminate the answers you know are incorrect.
  - The answer that is left is the correct answer.
- You can use the same type of logic to prove geometric things.

6.5 INDIRECT PROOF AND INEQUALITIES IN ONE TRIANGLE

- **Indirect Proof**
  - Proving things by making an assumption and showing that the assumption leads to a contradiction.
  - Essentially it is proof by eliminating all the other possibilities.

6.5 INDIRECT PROOF AND INEQUALITIES IN ONE TRIANGLE

- **Steps for writing indirect proofs**
  - Identify what you are trying to prove. Temporarily, assume the conclusion is false and that the opposite is true.
  - Show that this leads to a contradiction of the hypothesis or some other fact.
  - Point out that the assumption must be false, so the conclusion must be true.

6.5 INDIRECT PROOF AND INEQUALITIES IN ONE TRIANGLE

- Suppose you wanted to prove the statement "If $x + y \neq 14$ and $y = 5$, then $x \neq 9$." What temporary assumption could you make to prove the conclusion indirectly?

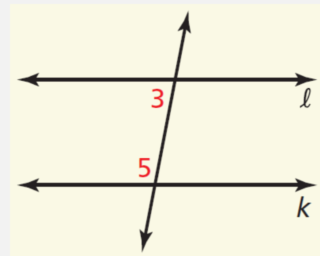
- Try #2

Assume x = 9

If x = 9, then x + y ≠ 14. 9 + 5 ≠ 14 → 14 ≠ 14. This is the contradiction

6.5 INDIRECT PROOF AND INEQUALITIES IN ONE TRIANGLE

- Write an indirect proof that if two lines are *not* parallel, then consecutive interior angles are *not* supplementary.
- **Given** Line ℓ is not parallel to line k.
- **Prove** $\angle 3$ and $\angle 5$ are not supplementary.



- Try #8

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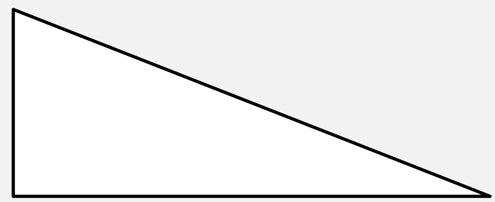
Assume temporarily that $\angle 3$ and $\angle 5$ are supplementary.
By the Converse of the Consecutive Interior Angles Theorem, line ℓ is parallel to line k.
This contradicts the given information.
So, the assumption that $\angle 3$ and $\angle 5$ are supplementary must be false, which proves that $\angle 3$ and $\angle 5$ are not supplementary.

6.5 INDIRECT PROOF AND INEQUALITIES IN ONE TRIANGLE

- Draw a scalene triangle
- Measure the sides
- Measure the angles
- What do you notice?



  - Smallest side opposite ________
  - Largest angle opposite _________

Smallest angle
Largest side

6.5 INDIRECT PROOF AND INEQUALITIES IN ONE TRIANGLE

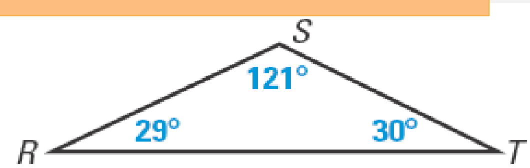
Big Angle Opposite Big Side Theorem

If one side of a triangle is longer than another side, then the angle opposite the longer side is larger than the angle opposite the shorter side.

Big Side Opposite Big Angle Theorem

If one angle of a triangle is larger than another angle, then the side opposite the larger angle is longer than the side opposite the smaller angle.

- List the sides in order from shortest to longest.
- Try #16



ST, RS, RT

6.5 INDIRECT PROOF AND INEQUALITIES IN ONE TRIANGLE

- Draw a triangle with sides 5 cm, 2 cm, and 2 cm.
- Draw a triangle with sides 5 cm, 2 cm, and 3 cm.
- Draw a triangle with sides 5 cm, 3 cm, and 3 cm.

Triangle Inequality Theorem

The sum of two sides of a triangle is greater than the length of the third side.

$AB + BC > AC$; $AB + AC > BC$; $BC + AC > AB$

Can’t be done, short side don’t touch
Can’t be done, forms a line
Can be done, isosceles triangle

6.5 INDIRECT PROOF AND INEQUALITIES IN ONE TRIANGLE

- A triangle has one side of 11 inches and another of 15 inches. Describe the possible lengths of the third side.
- Try #20, 24

$$11 + x > 15 \rightarrow x > 4$$
$$15 + x > 11 \rightarrow x > -4\ (already\ part\ of\ x > 4)$$
$$11 + 15 > x \rightarrow 26 > x$$

Combine 1st and 3rd: 4 < x < 26

Short cut: subtract to get smallest, add to get largest

6.6 INEQUALITIES IN TWO TRIANGLES

After this lesson…

- I can explain the Hinge Theorem.
- I can compare measures in triangles.
- I can solve real-life problems using the Hinge Theorem.

6.6 INEQUALITIES IN TWO TRIANGLES

- See Mr. Wright's demonstration with the meter sticks.
- How does the third side compare when there is a small angle to a big angle?

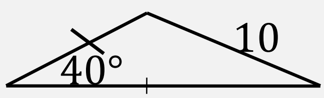
Use two meter sticks to demonstrate the Hinge Theorem

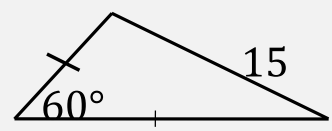
Have two meter sticks form two sides of the Δ and have the kids imagine the third side.

6.6 INEQUALITIES IN TWO TRIANGLES

Hinge Theorem

If 2 sides of one Δ are congruent to 2 sides of another Δ, and the included angle of the 1st Δ is larger than the included angle of the 2nd Δ, then the 3rd side of the 1st Δ is longer than the 3rd side of the 2nd Δ.

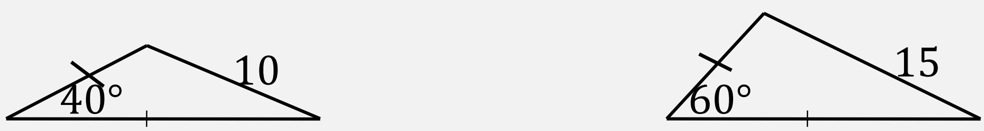




6.6 INEQUALITIES IN TWO TRIANGLES

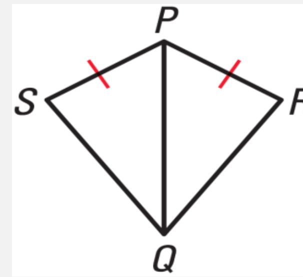
Converse of the Hinge Theorem

If 2 sides of one Δ are congruent to 2 sides of another Δ, and the 3rd side of the first is longer than the 3rd side of the 2nd Δ, then the included angle of the 1st Δ is larger than the included angle of the 2nd Δ.



6.6 INEQUALITIES IN TWO TRIANGLES

- If $PR = PS$ and $m\angle QPR > m\angle QPS$, which is longer, $\overline{SQ}$ or $\overline{RQ}$?

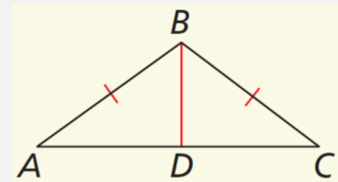


- If $PR = PS$ and $RQ < SQ$, which is larger, $m\angle RPQ$ or $m\angle SPQ$?

- Try #2, 6

6.6 INEQUALITIES IN TWO TRIANGLES

- Given: $\overline{AB} \cong \overline{BC}, AD > CD$
- Prove: $m\angle ABD > m\angle CBD$



Statements	Reasons

Try #10

$\overline{AB} \cong \overline{BC}, AD > CD$	(given)
$\overline{BD} \cong \overline{BD}$	(Reflexive)
$m\angle ABD > m\angle CBD$	(Converse of Hinge Theorem)

6.6 INEQUALITIES IN TWO TRIANGLES

- Two groups of joggers leave the same starting location heading in opposite directions. Each group travels 2 miles, then changes direction and travels 1 mile. Group A starts due north then turns 35° toward west. Group B starts due south then turns 25° toward east. Which group is farther from the start location? Explain your reasoning.

- Try #12

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Group B; The measure of the included angle for Group B is 155°, which is greater than the measure of the included angle for Group A. So, Group B is farther from camp.