EXPONENTIAL AND LOGARITHMIC FUNCTIONS

PRECALCULUS

CHAPTER 3

- This Slideshow was developed to accompany the textbook
  - *Precalculus*
  - *By Richard Wright*
  - *https://www.andrews.edu/~rwright/Precalculus-RLW/Text/TOC.html*
- Some examples and diagrams are taken from the textbook.

Slides created by
Richard Wright, Andrews Academy
rwright@andrews.edu

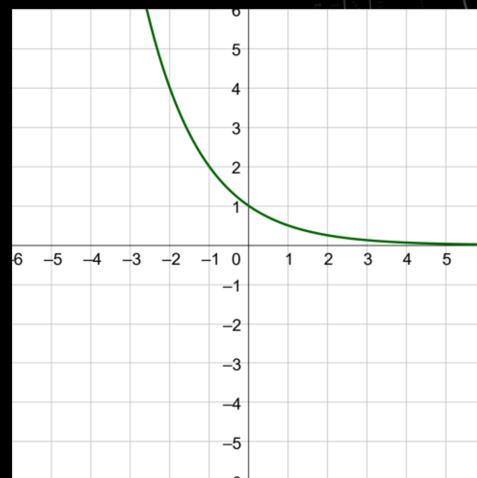
3-01 EXPONENTIAL FUNCTIONS

In this section, you will:

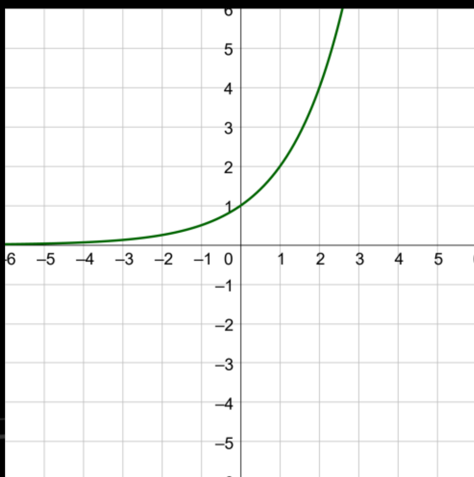
- Evaluate exponential functions with base *b*.
- Graph exponential functions with base *b*.
- Evaluate and graph exponential functions with base *e*.

3-01 EXPONENTIAL FUNCTIONS

- Exponential function
  - $y = a \cdot b^x$
  - a is initial amount (y-int)
  - b is base
  - x is exponent
- If $b > 1$
  - Exponential Growth
- If $0 < b < 1$
  - Exponential Decay



3-01 EXPONENTIAL FUNCTIONS



- Domain: All real numbers
- Range: $(0, \infty)$

- Horizontal Asymptote:
  - $y = 0$

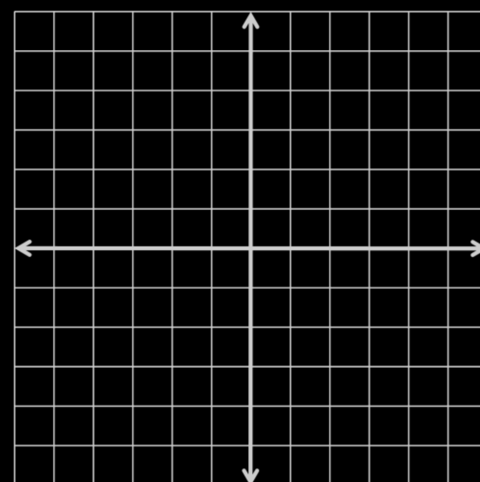
- y-intercept: (0, 1)

3-01 EXPONENTIAL FUNCTIONS

- Transformations
- $y = a \cdot b^{x-h} + k$
  - a vertical stretch
    - If a is negative, then reflected over x-axis
  - h moves right
  - k moves up
- Domain: All Real
- Range:
  - (k, ∞) if $a > 0$
  - $(-\infty, k)$ if $a < 0$
- HA: $y = k$
- y-int: $(0, a + k)$ if $h = 0$

3-01 EXPONENTIAL FUNCTIONS

- Graph by making a table
- Graph $y = 4^{-x} + 3$



- Decay
- HA: $y = 3$

3-01 EXPONENTIAL FUNCTIONS

- Exponential functions are one-to-one
  - Each *x* gives a unique *y*
- Solve $16 = 2^{x+2}$

$$16 = 2^{x+2}$$
$$2^4 = 2^{x+2}$$

Exponents must be equal

$$4 = x + 2$$
$$2 = x$$

3-01 EXPONENTIAL FUNCTIONS

- Solve $\left(\frac{1}{3}\right)^x = 81$

$$\left(\frac{1}{3}\right)^x = 3^4$$

$$\left(\frac{1}{3}\right)^x = \left(\frac{1}{3}\right)^{-4}$$

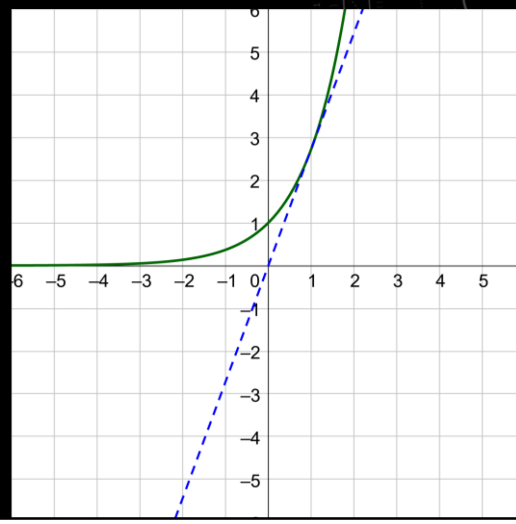
Exponents must be equal

$$x = -4$$

3-01 EXPONENTIAL FUNCTIONS

- Natural Base
  - $e \approx 2.718281828...$

- $f(x) = e^x$



Slope of any tangent line to e^x is e^x

3-01 EXPONENTIAL FUNCTIONS

- Compound Interest
- $A = P\left(1 + \frac{r}{n}\right)^{nt}$
- A = current amount
- P = principle (initial amount)
- r = yearly interest rate (APR)
- n = number of compoundings per year
- t = years

- Compounded Continuously
- $A = Pe^{rt}$
- $e = \left(1 + \frac{1}{n}\right)^{n}$
  - When $n \to \infty$

3-02 LOGARITHMIC FUNCTIONS

In this section, you will:

- Evaluate logarithmic functions with base *b*.
- Evaluate logarithmic functions with base *e*.
- Use logarithmic functions to solve real world problems.

3-02 LOGARITHMIC FUNCTIONS

- $f(x) = \log_b x$
  - "log base b of x"
- Logarithms are inverses of exponential functions
  - $y = \log_b x \leftrightarrow x = b^y$
- Logarithms are exponents!

3-02 LOGARITHMIC FUNCTIONS

- Evaluate
- Think "What exponent of the base gives the big number?"
- $\log_5 125$
- $\log_2 \frac{1}{64}$

Think $5^x = 125$
$5^3 = 125$
So $\log_5 125 = 3$

Think $2^x = \frac{1}{64}$
$2^{-6} = \frac{1}{64}$
So $\log_2 \frac{1}{64} = -6$

3-02 LOGARITHMIC FUNCTIONS

- Calculator
- LOG → $\log_{10}$ → log
- LN → $\log_e$ → ln

- Use your calculator to evaluate log 300

2.477

3-02 LOGARITHMIC FUNCTIONS

- Properties of Logarithms
- $\log_b 1 = 0$
- $\log_b b = 1$
- $\log_b b^x = x$
- If $\log_b x = \log_b y$, then $x = y$

- Simplify
- $\log_5 1$
- $\log_e e$
- $8^{\log_8 30}$

$$\log_5 1 = 0$$

$$\log_e e = \ln e = 1$$

$$8^{\log_8 30} = x$$

Rewrite as an exponential

$$\log_8 x = \log_8 30$$
$$x = 30$$

3-02 LOGARITHMIC FUNCTIONS

- Solve
- $\log_3(x^2 + 4) = \log_3 29$

Since logs are the same

$$x^2 + 4 = 29$$
$$x^2 = 25$$
$$x = \pm 5$$

3-03 PROPERTIES OF LOGARITHMS

In this section, you will:

- Use properties of logarithms to expand logarithmic expressions.
- Use properties of logarithms to condense logarithmic expressions.
- Use the change-of-base formula to evaluate logarithms.
- Graph logarithmic functions.

3-03 PROPERTIES OF LOGARITHMS

- Properties of Logarithms
  - Product Property: $\log_b(uv) = \log_b u + \log_b v$
  - Quotient Property: $\log_b\left(\frac{u}{v}\right) = \log_b u - \log_b v$
  - Power Property: $\log_b u^n = n \log_b u$

3-03 PROPERTIES OF LOGARITHMS

- Write each log in terms of $\ln 2$ and $\ln 5$.
- $\ln 10$
- $\ln \frac{5}{32}$

$$\ln 10$$
$$\ln(2 \cdot 5)$$
$$\ln 2 + \ln 5$$

$$\ln \frac{5}{32}$$
$$\ln \frac{5}{2^5}$$
$$\ln 5 - \ln 2^5$$
$$\ln 5 - 5 \ln 2$$

3-03 PROPERTIES OF LOGARITHMS

- Expand
- $\log 3x^2y$

- $\ln \frac{\sqrt{4x+1}}{8}$

$$\log 3 + \log x^2 + \log y$$
$$\log 3 + 2\log x + \log y$$

$$\ln(4x+1)^{\frac{1}{2}} - \ln 8$$
$$\frac{1}{2}\ln(4x+1) - \ln 8$$

3-03 PROPERTIES OF LOGARITHMS

- Condense
- $\frac{1}{3}\log x + 5\log(x-3)$
- $4\ln(x-4) - 2\ln x$

$$\log x^{\frac{1}{3}} + \log(x-3)^5$$
$$\log\left(x^{\frac{1}{3}}(x-3)^5\right)$$

$$\ln(x-4)^4 - \ln x^2$$
$$\ln\frac{(x-4)^4}{x^2}$$

3-03 PROPERTIES OF LOGARITHMS

- Condense
- $\frac{1}{5}(\log_3 x + \log_3(x-2))$

$$\frac{1}{5}(\log_3 x + \log_3(x-2))$$
$$\frac{1}{5}(\log_3 x(x-2))$$
$$\log_3\left(x(x-2)\right)^{\frac{1}{5}}$$
$$\log_3 \sqrt[5]{x(x-2)}$$

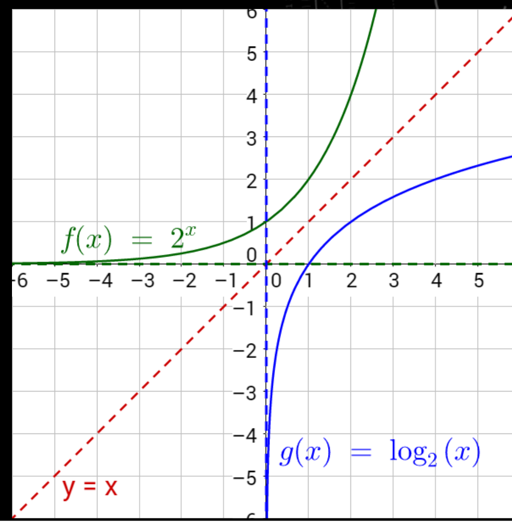
3-03 PROPERTIES OF LOGARITHMS

- Change-of-Base Formula
  - $\log_b c = \frac{\log_a c}{\log_a b}$
- Evaluate
- $\log_3 17$

$$\frac{\log 17}{\log 3} = 2.579$$

3-03 PROPERTIES OF LOGARITHMS

- Because logs are inverses of exponentials, the *x* and *y* is switched and the graph is flipped over the line *y* = *x*.
- $y = \log_b(x - h)$
  - Domain: $x > h$
  - Range: all real
  - VA: $x = h$
  - *x*-int: (*h* + 1, 0)

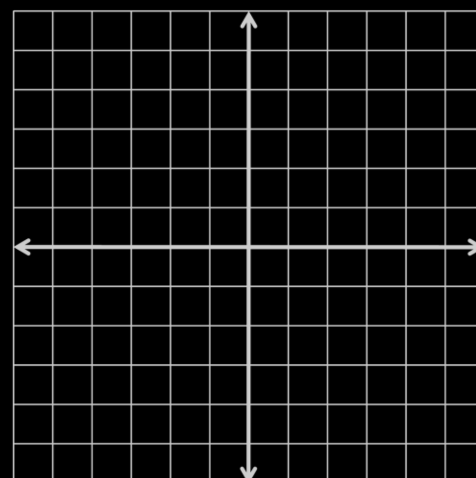


3-03 PROPERTIES OF LOGARITHMS

- To graph a logarithm
- Find and graph the vertical asymptote
- Make a table
  - Use change-of-base formula
    - $\log_b x = \frac{\log x}{\log b}$
  - Or use the logBASE function on some TI graphing calcs
    - MATH → logBASE

3-03 PROPERTIES OF LOGARITHMS

- Graph $y = \log_2(x + 1)$



Change-of-base gives $y = \log_2(x + 1) \rightarrow y = \frac{\log(x+1)}{\log 2}$

x	y
-1	Error
0	0
1	1
2	1.58
3	2
4	2.32
5	2.58
6	2.81
7	3

3-04 SOLVING EXPONENTIAL AND LOGARITHMIC EQUATIONS

In this section, you will:

- Use one-to-one property to solve exponential equations.
- Use one-to-one property to solve logarithmic equations.
- Solve general exponential equations.
- Solve general logarithmic functions.

3-04 SOLVING EXPONENTIAL AND LOGARITHMIC EQUATIONS

- Solve Exponential Equations
  - Shortcut Method
    - 1-to-1 method (rewrite with the same base)

- $\left(\frac{1}{5}\right)^x = 125$

$$\left(\frac{1}{5}\right)^x = 125$$
$$\left(\frac{1}{5}\right)^x = \left(\frac{1}{5}\right)^{-3}$$
$$x = -3$$

3-04 SOLVING EXPONENTIAL AND LOGARITHMIC EQUATIONS

- General Method
  - Take log of both sides
- $5 - 3e^x = 2$
- $6(2^{t+5}) + 4 = 11$

$$5 - 3e^x = 2$$
$$-3e^x = -3$$
$$e^x = 1$$
$$\ln e^x = \ln 1$$
$$x = 0$$

$$6(2^{t+5}) + 4 = 11$$
$$6(2^{t+5}) = 7$$
$$2^{t+5} = \frac{7}{6}$$
$$\log_2 2^{t+5} = \log_2 \frac{7}{6}$$
$$t + 5 = \log_2 \frac{7}{6}$$
$$t = -5 + \log_2 \frac{7}{6} \approx -4.778$$

3-04 SOLVING EXPONENTIAL AND LOGARITHMIC EQUATIONS

- $e^{2x} - 7e^{x} + 12 = 0$

$$(e^x - 3)(e^x - 4) = 0$$

$$e^x - 3 = 0 \qquad e^x - 4 = 0$$

$$e^x = 3 \qquad e^x = 4$$

$$\ln e^x = \ln 3 \qquad \ln e^x = \ln 4$$

$$x = \ln 3 \approx 1.099 \qquad x = \ln 4 \approx 1.386$$

3-04 SOLVING EXPONENTIAL AND LOGARITHMIC EQUATIONS

- Logarithmic Equations
- Shortcut Method
  - 1-to-1 Property

- $\ln x - \ln 3 = 0$

$$\ln x - \ln 3 = 0$$
$$\ln x = \ln 3$$
$$x = 3$$

3-04 SOLVING EXPONENTIAL AND LOGARITHMIC EQUATIONS

- General Method
  - Exponentiate both sides
- $6 + 3\ln x = 4$
- $\log_4 x + \log_4(x - 9) = 1$

$$6 + 3\ln x = 4$$
$$3\ln x = -2$$
$$\ln x = -\frac{2}{3}$$
$$e^{\ln x} = e^{-\frac{2}{3}}$$
$$x = e^{-\frac{2}{3}} \approx 0.513$$

$$\log_4 x + \log_4(x - 9) = 1$$
$$\log_4 x(x - 9) = 1$$
$$4^{\log_4 x(x-9)} = 4^1$$
$$x(x - 9) = 4$$
$$x^2 - 9x - 4 = 0$$
$$x = \frac{9 \pm \sqrt{9^2 - 4(1)(-4)}}{2(1)}$$
$$x = \frac{9 \pm \sqrt{97}}{2} \approx 9.424, \cancel{-0.424}$$

3-04 SOLVING EXPONENTIAL AND LOGARITHMIC EQUATIONS

- Graphical method
  - If the other methods don't apply
  - Make = 0
  - Find the x-int
- Solve $\log_2 x = \ln 2x$

$$\log_2 x - \ln 2x = 0$$

Graph and find x-int

$$x = 4.786$$

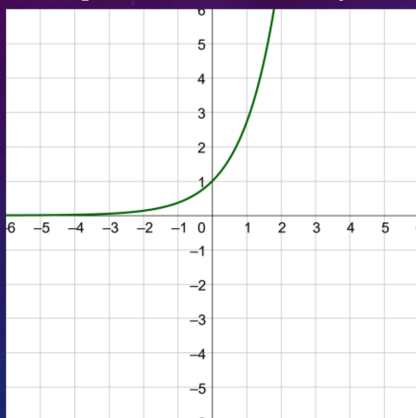
3-05 EXPONENTIAL AND LOGARITHMIC MODELS

In this section, you will:

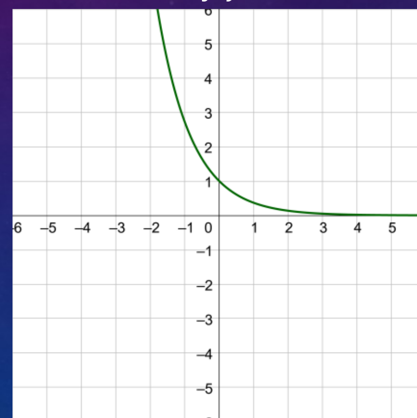
- Use exponential growth and decay models.
- Use the Gaussian model.
- Use the logistic growth model.
- Use logarithmic models.

3-05 EXPONENTIAL AND LOGARITHMIC MODELS

Exponential Growth $y = ae^{bx}$



Exponential Decay $y = ae^{-bx}$



3-05 EXPONENTIAL AND LOGARITHMIC MODELS

- Suppose a population growing according to the model $P = 800e^{0.03t}$ where t is in years.
- What is the initial size?
- How long to double?

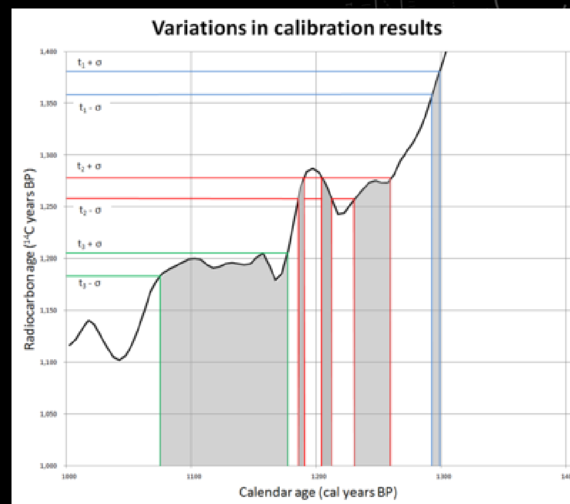
Let $t = 0$.

$$P = 800e^{0.03(0)} = 800$$

$$1600 = 800e^{0.03t}$$
$$2 = e^{0.03t}$$
$$\ln 2 = \ln e^{0.03t}$$
$$\ln 2 = 0.03t$$
$$t = 23.10\ yrs$$

3-05 EXPONENTIAL AND LOGARITHMIC MODELS

- Radioactive decay
  - $y = ae^{-bx}$
  - $A = A_0 e^{kt}$
  - Half-life
    - Time it takes for ½ of the material to decay



Very complicated, but we will use a simple model
The dating method depends on the initial conditions. These are not really known for prehistorical situations.

3-05 EXPONENTIAL AND LOGARITHMIC MODELS

- C^{14} has a half-life of 5700 years. If a sample starts with 3 g of C^{14}, how much will remain after 100 years?

Find decay constant

$$A = A_0 e^{kt}$$
$$1.5 = 3e^{k(5700)}$$
$$\frac{1}{2} = e^{k(5700)}$$
$$\ln\frac{1}{2} = \ln e^{k(5700)}$$
$$\ln\frac{1}{2} = k(5700)$$
$$k \approx -1.216 \times 10^{-4}$$

Find model

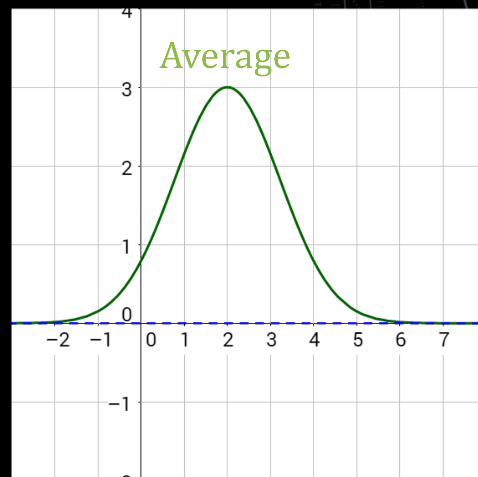
$$A = 3e^{(-1.216\times10^{-4})t}$$

Plug in 100 years

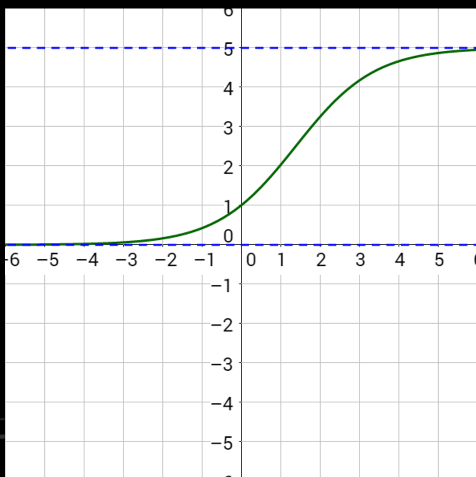
$$A = 3e^{(-1.216\times10^{-4})(100)} \approx 2.97\ g$$

3-05 EXPONENTIAL AND LOGARITHMIC MODELS

- Gaussian Model "The Curve"
- Normal Distribution
- $y = ae^{-\frac{(x-b)^2}{c}}$



3-05 EXPONENTIAL AND LOGARITHMIC MODELS



- Logistic Growth Model
- Used for population
- $y = \frac{a}{1+be^{-rx}}$

3-05 EXPONENTIAL AND LOGARITHMIC MODELS

- Logarithmic Models
  - $y = a + b \ln x$
  - $y = a + b \log x$
- Richter Scale
  - Earthquake magnitude
- Decibels
  - Loudness of sound